COMPSCI 323 - Computational Microeconomics

Recitation Week 1: Probability background

Jan. 13th, 2020

1 Probability model and combinatorics

A (simplified) probability model is defined as a tuple $(\Omega, \mathcal{F}, P)$, where:

- Ω is the set of **outcomes**, also called the sample space;
- $\mathcal{F}$ is a class of all subsets of Ω, or a collection of **events**;
- $P : \mathcal{F} \mapsto [0, 1]$ is a probability density function that satisfies the following axioms:
  - For any event A, $P(A) \geq 0$.
  - $P(\Omega) = 1$
  - For any pair of disjoint events A, A', $P(A \cup A') = P(A) + P(A')$

1.1

Roll a dice 4 times. What is the probability that all four numbers are different?

Solution Let the sample space be

$$\Omega = \{(1, 1, 1, 1), (1, 1, 1, 2), \ldots, (6, 6, 6, 6)\}.$$

The event E where all four rolls are different would look like:

$$E = \{(1, 2, 3, 4), (1, 2, 3, 5), \ldots, (6, 5, 4, 3)\}.$$

Since all of the outcomes in Ω are equally likely, the probability we are looking for is $P(E) = \frac{|E|}{|\Omega|} = \frac{6 \cdot 5 \cdot 4 \cdot 3}{6^4}$.

1.2

Shuffle a deck of cards.

- What is the probability that an Ace ends up at the top of the deck?
- What is the probability that all Aces are together in the deck?

Solution to first subproblem $\frac{(4)(51!)}{52!}$

Solution to second subproblem Imagine the four Aces as one card to easily compute the number of arrangements in which the Aces end up together. Also keep in mind that the Aces can be arranged differently between themselves. It follows that the probability we are looking for is $\frac{(4!)(49!)}{52!}$.

1.3

Shuffle a deck of cards and deal 13 cards to each of the 4 players. What is the probability that each person gets an Ace?

Solution Suppose, as we did in problem 1.2, that the sample space is composed of all possible arrangements of the 52 cards. The size of the sample space is then 52!.
Now, it only remains to count the number of arrangements in which every player is dealt one Ace. For each player, there are 13 positions where an Ace can be placed. Also, the four different Aces can be arranged differently between themselves. Finally, the rest 48 cards can be arranged in whatever way, so the total number of arrangements that result in every player getting an Ace is $(13^4)(4!)(48!)$.
The final solution is $\frac{(13^4)(4!)(48!)}{52!}$.

A classmate of ours approached this problem differently. He essentially divided the problem into three separate random experiments: dealing 13 cards out of the 52 cards to the first player, dealing 13 cards out of the remaining 39 cards to the second player, and dealing 13 cards out of the remaining 26 cards to the third player. It is only when each of these three independent experiments hands out exactly one Ace to the player that all four players will end up having an Ace. Note that the probabilities from the three experiments can be multiplied together because they are independent. This interesting line of reasoning leads to the following formulation of the equivalent solution:

$$\frac{4 \cdot \binom{48}{12}}{\binom{52}{13}} \cdot \frac{3 \cdot \binom{36}{12}}{\binom{39}{13}} \cdot \frac{2 \cdot \binom{24}{12}}{\binom{26}{13}}.$$

2 Conditional probability

2.1

Vince throws three coins and reveals two of them to you. Suppose Vince will show as many heads as he can. If the two coins that Vince reveals are heads, what is the probability that the other coin is also a head?

Solution Let the sample space be

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Then, the conditional probability we are looking for is

$$\begin{aligned} P(\text{hidden is H}|\text{revealed are HH}) &= \frac{P(\text{hidden is H} \cap \text{revealed are HH})}{P(\text{revealed are HH})} \\ &= \frac{1/8}{4/8} \\ &= \frac{1}{4} \end{aligned}$$

3 Random variables

A **random variable** is a function from a sample space Ω to the real numbers $\mathbb{R}$.

A **discrete** random variable X takes a countable number of different values x_i.

- Probability distribution of its values is described by a **probability mass function**

$$p(x_i) = P(X = x_i)$$

- Expectation is then defined as

$$E[X] = \sum_i x_i p(x_i)$$

A **continuous** random variable takes on any value on some intervals.

- Probability distribution described by a **probability density function**

$$P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x)dx$$

- **Cumulative distribution function** is defined as

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$$

  Note that $F'(x) = f(x)$.

- Expectation defined as

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)dx$$

3.1

Suppose a post office informs you that your mail will arrive between 2 PM and 4 PM today. Assume that the time of the delivery is uniformly distributed in this interval.

- What is the expected time of the mail delivery?
- What is the probability that the mail arrives between 3 PM and 3:20 PM today?
- At 3:30 PM, the mail still hasn't arrived. What is the probability that it arrives in the next 10 minutes?
- Suppose you are also expecting a pizza delivery between 2:15 PM and 3:15 PM. Assume that the pizza delivery time also follows a uniform distribution in this interval. Also assuming the independence between the two deliveries, what is the probability that you have both the mail and the pizza by 3 PM today?

Solution to first subproblem Let the random variable $M \sim U[0, 120]$ represent the time of mail delivery (in minutes since 2 PM). Then, the probability density function for M is given by the function $f(x) = \frac{1}{120}$, for $0 \leq x \leq 120$. Now, by definition of expectation for a continuous random variable, we have

$$\begin{aligned} E[M] &= \int_{-\infty}^{\infty} x \cdot f(x) dx \\ &= \int_{0}^{120} x \cdot \frac{1}{120} dx \\ &= \frac{1}{120} \left[\frac{x^2}{2} \right]_0^{120} \\ &= 60. \end{aligned}$$

Solution to second subproblem

$$\begin{aligned} P(60 \leq M \leq 80) &= \int_{60}^{80} \frac{1}{120} dx \\ &= \frac{1}{120} [x]_{60}^{80} \\ &= \frac{1}{6} \end{aligned}$$

Solution to third subproblem

$$\begin{aligned}
P(M \leq 100 | M \geq 90) &= \frac{P(90 \leq M \leq 100)}{P(M \geq 90)} \\
&= \frac{10/120}{30/120} \\
&= \frac{1}{3}
\end{aligned}$$

Solution to fourth subproblem Let the random variable $P \sim U[15, 75]$ represent the time of pizza delivery (in minutes since 2 PM). Then,

$$\begin{aligned}
P(M \leq 60, P \leq 60) &= P(M \leq 60)P(P \leq 60) \\
&= \frac{60}{120} \cdot \frac{45}{60} \\
&= \frac{3}{8}.
\end{aligned}$$

Note that the first equality holds because M and P are independent.