Problem Solving

Definition of a Problem

- A set of given information
  - description of the problem.
- A set of operations
  - permissible moves or actions.
- A goal
  - description of what constitutes a solution.

Polya’s Method

“How to Solve It”

1. Understand the problem
2. Devise a plan
3. Carry out the plan
4. Look back

Understand the problem

- Read the problem carefully - if you don’t know what needs to be solved, then you will likely get an incorrect solution
  - try stating the problem in your own words or explaining the problem to someone else
- Identify the unknown quantities
- Identify the given (known) quantities
- Identify the given conditions
- Identify any constraints
- Introduce suitable notation for unknowns
- It may help to draw a picture, chart or diagram

Devise a plan

- Establish relation between known and unknown quantities
  - simple case - an algebraic expression or equation
- If you do not find an obvious relation, try the following:
  - Divide the problem into subproblems you already know how to solve
  - Compare the problem to a similar problem you already know how to solve
  - Try to solve an easier version of the problem first
  - Guess the solution and work backwards
  - Introduce intermediate quantities
  - Look for a pattern

Carry out the plan

- Perform necessary computations
- Solve any equations
- Find a solution
- Check each step!
  - you should be able to prove that each step of your plan is correct
### Look back
- Examine the solution obtained
- Check the results in the original problem
- Check to see if you used all your information
- Does your answer make sense?

### Sum of consecutive integers

The sum of two consecutive integers is 23. Find the integers.

1. Understand the problem
   - known: the sum is 23
   - unknowns: 
x
   - the first consecutive integer
x+1
   - the second consecutive integer

2. Devise a plan
   \[ x + (x+1) = 23 \]

3. Carry out the plan
   \[ x = 11, \ x + 1 = 12 \]

4. Look back
   \[ 11 + 12 = 23 \]

### Sum of consecutive odd integers

The sum of three consecutive odd integers is 57. Find the integers.

- This problem is similar to the previous one
  - three unknowns (instead of two)
  - integers are odd
- Represent the first integer by \( x \)
  - the next consecutive odd integer is \( x + 2 \)
  - the third consecutive odd integer is \( x + 4 \)
- Relate knowns to unknowns: \[ x + (x+2) + (x+4) = 57 \]
- Solve and check

### Dividing The Spoils

After gathering 770 chestnuts, the three little girls divided them up so that their amounts were in the same proportion as their ages. As often as Mary took four chestnuts, Nellie took three, and for every six that Mary received, Susie took seven. How many chestnuts did each girl get?

### Solving the problem

- After gathering 770 chestnuts, the three little girls divided them up so that their amounts were in the same proportion as their ages. As often as Mary took four chestnuts, Nellie took three, and for every six that Mary received, Susie took seven. How many chestnuts did each girl get?
- Notice that Mary is related to both Susie and Nellie
  - Let \( c \) = number of chestnuts Mary takes
  - Then \[ 3c/4 \] = number of chestnuts Nellie takes
  - \[ 7c/6 \] = number of chestnuts Susie takes
  - As there are 770 chestnuts in all, \[ c + (3c/4) + (7c/6) = 770 \]
- Solve and check

### Problem solving is easier with practice

“Solving problems is a practical art, like swimming, or skiing, or playing the piano; you can learn it only by imitation and practice ... if you wish to learn swimming you have to go into the water, and if you wish to become a problem solver you have to solve problems.”

G. Polya
### Apples and Pears

- Tom has three boxes with fruits in his barn: one box with apples, one box with pears, and one box with both apples and pears. The boxes have labels that describe the contents, but none of these labels is on the right box. How can Tom, by taking only one piece of fruit from one box, determine what each of the boxes contains?

### A Strange Liar

- Richard is a strange liar. He lies on six days of the week, but on the seventh day he always tells the truth. He made the following statements on three successive days:
  - Day 1: “I lie on Monday and Tuesday”
  - Day 2: “Today is Thursday, Saturday, or Sunday”
  - Day 3: “I lie on Wednesday and Friday”

On which day does Richard tell the truth?

### The Round Table

- Helen and her husband invited their neighbors - two couples - for a dinner at their home. The six of them sat at a round table. Helen tells you the following:
  - Victor sat on the left of the woman who sat on the left of the man who sat on the left of Anna.
  - Esther sat on the left of the man who sat on the left of the woman who sat on the left of the man who sat on the left of my husband.
  - Jim sat on the left of the woman who sat on the left of Roger.
  - I did not sit beside my husband.

What is the name of Helen’s husband?

### The farmer, the goose, the grain, and the fox

- Imagine that you are a farmer with a goose, a fox, and some grain. You have to get across a river with all your belongings. However, you can only take one item on the boat at a time. You cannot leave the goose with the grain, or he will eat the grain. You cannot not leave the fox with the goose, or the fox will eat the goose. How can you get all your belongings safely to the other side?

### Missionaries and Cannibals

- There are three missionaries and three cannibals on one side of a river. They need to cross to the other side. The boat to carry them across can only take two at a time. If at any time the cannibals on one shore outnumber the missionaries, the cannibals will eat the missionaries. How can all the people safely cross the river?

### Jane, Jean and Joan

- Joan and Jane are sisters. Jean is Joan’s daughter and 12 years younger than her aunt. Joan is twice as old as Jean. Four years ago, Joan was the same age as Jane is now, and Jane was twice as old as her niece. How old is Jean?