Trees

- Set of elements or nodes
  - one node as root
  - every other node has a single predecessor (parent) and zero or more successors (children)
  - nodes without successors are called leaves
- Nodes are connected by edges
- The depth of a node is the number of edges on the path to the root
- The height of the tree is the number of edges on the longest path from root to any leaf
- A complete tree has all leaves at same depth, all other nodes have the same number of children

Example

Questions

- What is the height of the tree?
- What is the depth of node n?
- What is the depth of node p?
- Is the tree complete?

Subtrees

- The subtree rooted at x is the tree of all descendants of x that has node x as root

Questions
Binary trees

- A binary tree is a structure defined on a finite set of nodes that either
  - contains no nodes
  - is composed of three disjoint sets of nodes
    - a root node
    - a binary tree called its left subtree
    - a binary tree called its right subtree

Left and right subtrees

- In a binary tree
  - the left subtree of x is the tree rooted at the left child of x
  - the right subtree of x is the tree rooted at the right child of x

Questions

- In a complete binary tree of height 3,
  - how many nodes are in the tree?
  - how many leaves are in the tree?
- In a complete binary tree of height 4,
  - how many nodes are in the tree?
  - how many leaves are in the tree?
- How many nodes in a complete binary tree of height h?
- How many leaves in a complete binary tree of height h?
Complete binary tree of height 3
- 15 nodes
- 8 leaves

Complete binary tree of height 4
- 31 nodes
- 16 leaves

Binary trees
In a binary tree of height h
- the total number of nodes is at most
  \[2^0 + 2^1 + \ldots + 2^h = 2^{h+1} - 1\]
- the number of leaf nodes is at most \(2^h\)

If the tree is complete
- there are exactly \(2^{h+1} - 1\) nodes
- there are exactly \(2^h\) leaves

Decision trees
- A decision tree begins with a decision you need to make
  - start with an initial decision node
  - ask a question
- Structure for investigating options and the possible outcomes of choosing those options
  - result of a decision can be another decision
  - outcome is a terminal node
- Tree should have unique paths from the decision node to each of the terminal nodes.
  - help you to choose between several courses of action

A decision tree
Would you prefer a humanitarian?

- yes
  - Would you like to read about Einstein?
    - yes
      - He received the Physics Prize in 1921.
    - no
      - Try the Medicine Prize in 1962.
  - no
    - Look up the Peace Prize in 1991.

- no
  - Would you like to read about a scientist?
    - yes
      - Try the Prize in Literature, 1970.
    - no

A decision tree for insertion sort
sort three elements a,b,c

- \(b \leq c\)
  - yes
    - \(a \leq b\)
    - no
  - no
    - \(a \leq c\)
    - yes
      - \(b \leq c\)
      - no
    - no
      - \(a \leq b\)
      - yes
        - \(c \leq a\)
        - no
      - no

Try the Prize in Literature, 1970.
Decision-tree model

- Decision trees are used to model comparison sorts
  - insertion sort, mergesort, quicksort
- Each possible permutation of n elements must appear as a leaf in the tree
  - at least n! leaves
- Worst-case number of comparisons = length of longest path from root to leaf
- Insertion sort worst case:
  - number of comparisons: 0+1+ ... + n-1 = ½ (n²-n)
- Length of longest path in decision tree for 4 elements = ½ (4²-4) = 6