B - Probability and Graphs

Probability

\[ S = \text{set of all possible outcomes} = \{HH, HT, TH, TT\} \]

\[ s = \text{single outcome} = HH \]

\[ A, B = \text{sets of outcomes} = \{HH, HT\}, \{TH, TH\} \]

Flip two coins at the same time

1. \( Pr[\{s\}] = 1 \)
2. \( Pr[A] \geq 0 \) \( \forall A \subseteq S \)
3. \( Pr[A \cup B] = Pr[A] + Pr[B] \) if \( A \cap B \) mutually excl.

\[ = Pr[A] + Pr[B] - Pr[A \cap B] \]

Uniformly Distributed if

\[ \forall s, \ Pr[\{s\}] = \frac{1}{|S|} \]
Independent

\[ \Pr \{ A \land B \} = \Pr \{ A \} \cdot \Pr \{ B \} \]

\[ \frac{1}{2} \quad \frac{1}{2} \cdot \frac{1}{2} \]

A = \{ HH, HT \}
B = \{ HH, TH \}

A = \{ HH \} \text{ welded}
B = \{ HH \} \text{ together}
Random Variables

Random variable $X$ is a function from a set of all possible outcomes to the reals.

*of heads

\[
\begin{align*}
\text{HH} & \rightarrow 2 \\
\text{HT} & \rightarrow 1 \\
\text{TH} & \rightarrow 1 \\
\text{TT} & \rightarrow 0
\end{align*}
\]

\[\Pr \xi X \geq 3 = \frac{1}{4}\]

\[\Pr \xi X = \frac{1}{3}\]

Prob density fn: \( f(x) = \Pr \xi X = \frac{1}{3} \)

Prob distribution fn: \( F(x) = \Pr \xi X \leq x \frac{1}{3} \)

\[
F(x) = \int_{-\infty}^{x} f(y) \, dy
\]
Roll 2 dice

\( X = \text{max value on a die} \)

\( Y = \text{min value on a die} \)

\[ \Pr \{ \frac{1}{2}X = 6 \text{ and } Y = 1 \} = \frac{1}{36} \cdot \frac{1}{2} = \frac{1}{18} \]

\[ = \Pr \{ \frac{1}{2}X = 6 \} \cdot \Pr \{ Y = 1 \} \]

\( X = \text{value on die 1} \)

\( Y = \text{value on die 2} \)

\[ \Pr \{ \frac{1}{2}X = x \text{ and } Y = y \} = \Pr \{ \frac{1}{2}X = x \} \cdot \Pr \{ Y = y \} \]
Expected Value

\[ E[X] = \sum_{x} x \cdot \Pr \{ X = x \} \]

Weighted mean

\[ E[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} \]

\[ \frac{21}{6} = 3 \frac{1}{2} \]

\[ E[X+Y] = E[X] + E[Y] \]

Linearity of expectation

\[ E[\Theta(f(n) \cdot X)] = \Theta(f(n) \cdot E[X]) \]
Graphs

Graph has nodes & edges \( G = (V, E) \)

\[ \begin{array}{c}
\emptyset \text{ not ok} \quad \emptyset \text{ ok} \\
\hline
\text{Graph} \quad \text{Graph} \quad \text{Not Graph} \quad \text{Not graph}
\end{array} \]

Vertex can be thought of as a point

Edge **has** to connect 2 vertices

- Directed graph where edges go from one vertex to another
- Undirected graph where edges go both ways.
V is adjacent to U if an edge connects U to V.

Degree of vertex U:
# of edges incident on U = 3

In-degree = 3
Out-degree = 3

Directed Graph
Isolated
if degree = 0 in undir
if in and out-degree = 0 in dir.
$u$ and $v$ are connected if there is a path from $u$ to $v$.

Cycle exists when a vertex has a path to itself.

Simple path contains no cycles.
negative cycle

not neg cycle
Strongly Connected Component
All vertices in the cc. have edges to all other vertices.

Complete Graph
All vertices have edges to each other.

One graph (A) is isomorphic to graph (B) if A's vertices can be reordered to create B.
Bipartite Graph

\[ V_1 = \{1, 2, 3, 4\} \]
\[ V_2 = \{5, 6, 7, 8\} \]

Tree

Undirected, acyclic, connected graph

\[ |V| = |E| + 1 \]

Forest

Undirected, acyclic graph

\[ |V| \geq |E| + 1 \]