Information and Instructions

- Open-source and citation. This is an open-source exam, on the condition that the sources be properly cited. Proper citation reflects critical selection and use of existing knowledge and eliminates intentional or unintentional plagiarism

  You may bring in books and notes. The citation includes the page numbers as well as the book titles and authors. You may also search on the Internet, cite the exact web addresses.

- No-live communication is allowed with anyone else, in person or via any electronic media.

- Problem set and grading. Each individual selects **4** out of the 5 problems as the primary ones for the exam grading. Please mark with X the numbers for the 4 problems you choose as the primary.

  (1) ______ (2) ______ (3) ______ (4) ______ (5) ______

  The additional problem, if you wish to submit as well, will not get a score higher than the average of the primary ones.

- The table of contents below is to help you glance at and navigate the problem set. The elaborated sub-items in each problem are intended to help you organize your thoughts.

- Concise answers and arguments are preferred; irrelevant and incorrect comments are subject to point deduction.

- The grading process is blind to individual names. Please mark every page of your answers with the ID provided to you.

- Upon completion, please submit your answers and return the problem description to the DGS office.

  If you decide not to continue and finish the exam, please inform the DGS office so, on the returned problem set.

Good luck !

Contents

The problems in this exam, reaarding some of the basic in Numerical Computing, are put into a particular context of geometric projections, given the broader use of the inner product in modern data analysis and modeling.

Denote by $\mathbb{C}_m[a, b]$ the set of real-valued continuous functions over $[a, b]$, $b > a$. Define

$$\langle f, g \rangle = \int_a^b f(x)\, q(x) dx, \qquad \forall\, f,\, g \in \mathbb{C}_m[a, b]. \tag{1}$$

1 Geometric projections in a polynomial space

Denote by $\mathbb{P}_m[a, b]$ the set of polynomials of degree equal to or less than m over $[a, b]$, $m > 0$, with real valued coefficients.

1.1 Questions(1,2,3)

1. Confirm with Yes, or reject with No, each of the following statements, or provide a brief answer when neither of the binary answers is adequate.

   (a) The number of members in $\mathbb{P}_m[a, b]$ may be arbitrarily large, but finite.

   (b) $\mathbb{P}_m[a, b]$ is a vector space of dimension $(m + 1)$ and a subspace of $\mathbb{C}[a, b]$.

   (c) The functional in (1) is bilinear in its arguments.

   (d) By (1), when associated with the real field $\mathbb{R}$, $\langle p - q, p - q \rangle$ is non-negative but $\langle p - q, p - q \rangle^{1/2}$ is not necessarily a distance function in $\mathbb{P}_m[a, b]$.

2. Provided with N polynomials $p_j(x) \in \mathbb{P}_m[a, b]$, $j = 1, 2, \cdots, N$. Describe a method to calculate the pairwise projection coefficients

$$\langle p_i, p_j \rangle, \quad \forall\, i,\, j \in \{1, 2, \cdots, N\}.$$

   Specify clearly the particular expression you use for the polynomials.

3. Apply the Gram-Schmidt process to find the orthogonal polynomials in $\mathbb{P}_m[a, b]$ with respect to the inner product of (1), assuming a function/subroutine for calculating the inner product is available.

2 Function approximation with piecewise polynomials

2.1 Questions(4)

4. Describe clearly and briefly a method that approximates a smooth function in $\mathbf{C}[a, b]$ with piecewise polynomials.

3 Data fitting with piecewise polynomials

3.1 Questions(5)

5. Provided M data points (x_i, y_i), $i = 1, 2, \cdots, M$, with x_i distributed in N equispaced sub-intervals of $[a, b]$, where $M >> N$, and there are at least two points over each and every sub-interval. Describe a model and a method to fit the discrete data with a cubic spline made of N pieces over the N sub-intervals.