Complexity Qual Spring 2020
(provide proofs for all answers)
(120 Points Total)

Problem 1: Formal Languages (30 Points Total):

For each of the following languages, state and prove whether: (i) the language is regular, (ii) the language is context-free but not regular, or (iii) the language is non-context-free.

Problem 1a (10 Points): $L_a = \{0^n 1^n | n \geq 0\}$ (that is, the set of all binary strings consisting of 0's and then 1's where the number of 0's and 1's are equal).

Problem 1b (10 Points): $L_b = \{w \in \{0,1\}^* | w$ has an even number of 0's and at least two 1's$\}$ (that is, the set of all binary strings with an even number, including zero, of 0's and at least two 1's).

Problem 1c (10 Points): $L_c = \{0^n\#0^{2n}\#0^{3n} | n \geq 0\}$ (that is, the set of all strings of the form 0 repeated n times, followed by the # symbol, followed by 0 repeated $2n$ times, followed by the # symbol, followed by 0 repeated $3n$ times).

Problem 2: NP-Hardness (30 Points Total):

There are a set of n people who are the vertices of an undirected graph G. There's an undirected edge between two people if they are enemies.

Problem 2a (15 Points): The people must be assigned to the seats around a single large circular table. There are exactly as many seats as people (both n). We would like to make sure that nobody ever sits next to one of their enemies. Consider the problem of determining whether this is possible. Can it be done in polynomial time or is it NP-hard? Prove it.

Problem 2b (15 Points): The same problem, except now we want to make sure that nobody ever sits directly across (opposite) from one of their enemies. (It is now OK if enemies sit next to each other.) You may assume n is even, so there is exactly one seat directly across from each seat.Again, can it be done in polynomial time or is it NP-hard? Prove it.
Hint: One of the problems is related to a well-known graph problem that is know to be solvable in P, and one of these problems is somehow related to a famous NP-Hard graph path problem.

Problem 3: Relations between Space and Time Complexity (30 Points Total):

Let P be be the set of all languages accepted by Turing Machines that, with input size n, take no more than time polynomial in n.
Let DSPACE(n) be the set of all languages accepted by deterministic Turing Machines that with input size n, take no more than $O(n)$ space.
Let EXPTIME be the set of all languages accepted by deterministic Turing Machines that, with input size n, take no more than 2^{cn} time, for some constant c.

Problem 3a (15 Points): Show that if DSPACE(n) = EXPTIME, then there is a language in DSPACE(n) that is not in P.

Problem 3b (15 Points): Show that if DSPACE(n) = P, then there is a language in EXPTIME that is not in DSPACE(n).

Problem 4: Decidability and Undecidability (30 Points Total):

Assume an encoding of each Turing Machine descriptions as a unique binary string, and let M_α be the Turing Machine encoded by binary string α. Let $A(x, y)$ be the function:
$A(x, y) = y + 1$ if $x = 0$,
$A(x, y) = A(x - 1, 1)$ if $x > 0$ and $y = 0$
$A(x, y) = A(x - 1, A(x, y - 1))$ if $x > 0$ and $y > 0$.

Problem 4a (10 Points): Is $A(x, y)$ computable or not? Give a proof.

Problem 4b (10 Points): Let L_1 be the set of pairs (α, x) such that Turing Machine M_α halts in at most $A(|x|, |x|)$ steps on input x (where $|x|$ is the length of input string x). Is L_1 decidable or is A undecidable? Give a proof.

Problem 4c (10 Points): Let L_2 be the set of strings α such that Turing Machine M_α halts on some input. Is L_2 decidable or is L_2 undecidable? Give a proof.