Bellman-Ford Algorithm

Computes single source shortest paths even when some edges have negative weight.

The algorithm detects if there are negative cycles reachable from s.

If there are no such negative cycles, it returns the shortest paths.

The algorithm has two parts:

Part 1: Computing Shortest Paths Tree:

$|V| - 1$ iterations, iteration i computes the shortest path from s using paths of up to i edges.

Part 2: Checking for Negative Cycles.

Bellman-Ford (G, w, s)

1. For all $v \in V$ do

   1.1 $d[v] \leftarrow \infty$;
   1.2 $\pi[v] \leftarrow NIL$;

2. $d[s] = 0$;

3. For $i \leftarrow 1$ to $|V| - 1$ do

   3.1 For all $(u, v) \in E$ do
   3.1.1 If $d[v] > d[u] + w(u, v)$ then
   3.1.1.1 $d[v] \leftarrow d[u] + w(u, v)$;
   3.1.1.2 $\pi[v] \leftarrow u$;

4. For all $(u, v) \in E$ do

   4.1 If $d[v] > d[u] + w(u, v)$ then return FALSE;

5. return TRUE

Run Time

Theorem 1. *The run time of the algorithm is $O(V \times E)$.*

Proof.

The initialization (1) takes $O(V)$.

The path creation (3) takes $O(V \times E)$.

The negative cycle detection (4) takes $O(E)$. $\square$

Correctness

Theorem 2. *Assume that G contains no negative cycles reachable from s then the algorithm computed shortest paths for all vertices of G.*

Proof. Fix a vertex $u \in V$, we prove that the algorithm computes a shortest path from s to u.

Let $P = v_0, v_1, ..., v_k$, where $v_0 = s$ and $v_k = u$ be a shortest path from s to u.

Since there are no negative cycles P is a simple path, $k \leq |V| - 1$.

We prove by induction on i that after the i-th iteration of the (3) loop, the algorithm computed the shortest path for v_i.

The hypothesis holds for $v_0 = s$.

Assume that it holds for $j \leq i - 1$. After the i-th iteration

$$d[v_i] \leq d[v_{i-1}] + w(v_{i-1}, v_i)$$

which is the shortest path from s to v_j, since P is a shortest path from s to v_k, and this is the distance between s to v_j on that path.

$\square$

Theorem 3. *The algorithm returns TRUE if there are no negative cycles reachable from s, otherwise it returns FALSE.*

Proof. Assume that there are no negative cycles reachable from s, then by the previous theorem, the algorithm returns a shortest path tree, and $d[v]$ is the weight of the shortest path to s.

Thus, all inequalities in 4.1 don't hold.

Assume that there is a negative weight cycle $v_0, ..., v_k$ reachable from s ($v_0 = v_k$).

Since the path is reachable from s the values $d[v_i]$ are defined.

$\sum_{i=1}^{k} d[v_{i-1}] = \sum_{i=1}^{k} d[v_i]$ and

$$\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0.$$

Thus,

$$\sum_{i=1}^{k} d[v_{i-1}] > \sum_{i=1}^{k} d[v_i] + \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

So there must be an i such that

$$d[v_{i-1}] > d[v_i] + w(v_{i-1}, v_i)$$

and the algorithm returns FALSE $\square$