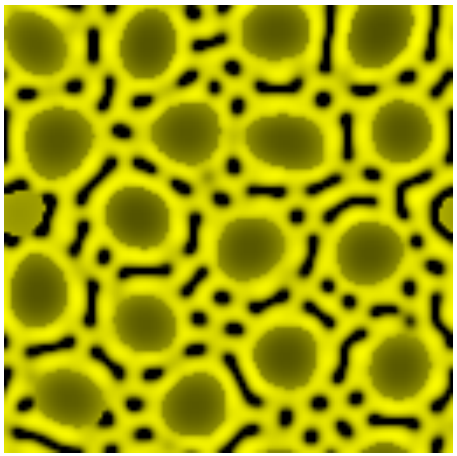
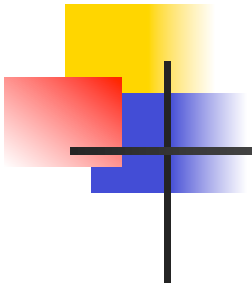
Pattern Formation by Reaction-Diffusion

Sertan Girgin

Ahmet Saçan

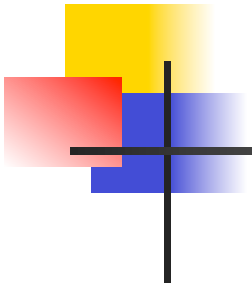




Game-plan

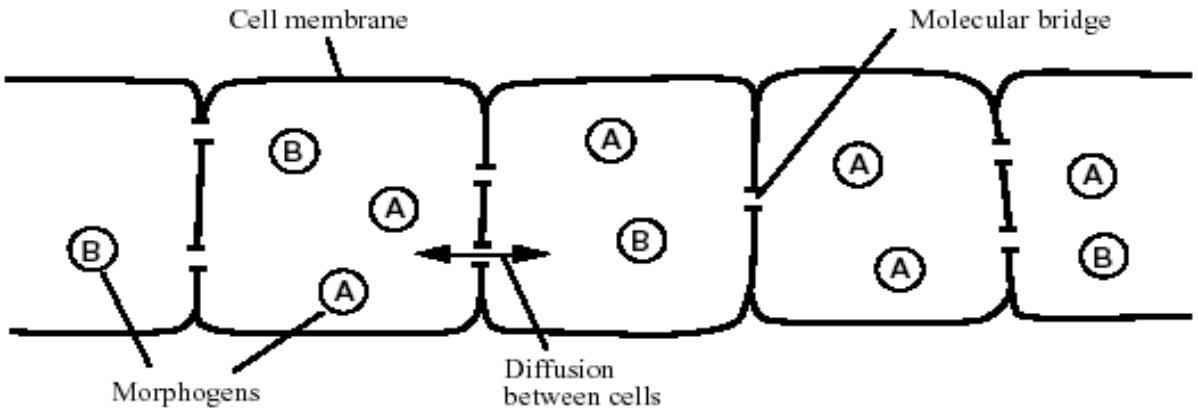
- Reaction-Diffusion defined.
- Mathematical Model
- Solution to RD
- Simulations
- Parameters
- History of RD: models & applications

Reaction Diffusion (RD)



- A chemical mechanism for pattern formation.
- First described by Alan Turing (1952).
- *Two* chemicals diffusing across a *surface* and reacting with one another can form stable patterns of chemical concentration.

RD in a line of cells



• The amount of chemical *a* in a cell changes based on the quantity of the chemicals *a* and *b* are already in the cell.
• If a particular cell has a higher concentration of chemical *b* than its neighbors, then that cell's concentration of *b* will decrease over time by diffusion to its neighbors. Likewise, if the concentration of *b* is at minimum at a particular place along the row of cells, then more of *b* will diffuse from adjacent cells to this cell to raise the concentration of *b* at that cell.

Mathematical Model

$$\frac{\partial a}{\partial t} = D_a \nabla^2 a + F(a,b)$$

$$\frac{\partial b}{\partial t} = D_b \nabla^2 b + G(a,b)$$

Analytical Solution?

- Closed-form solution: difficult/ impossible (except when F,G very simple).

- Therefore,
  - Discretize and
  - Solve numerically.

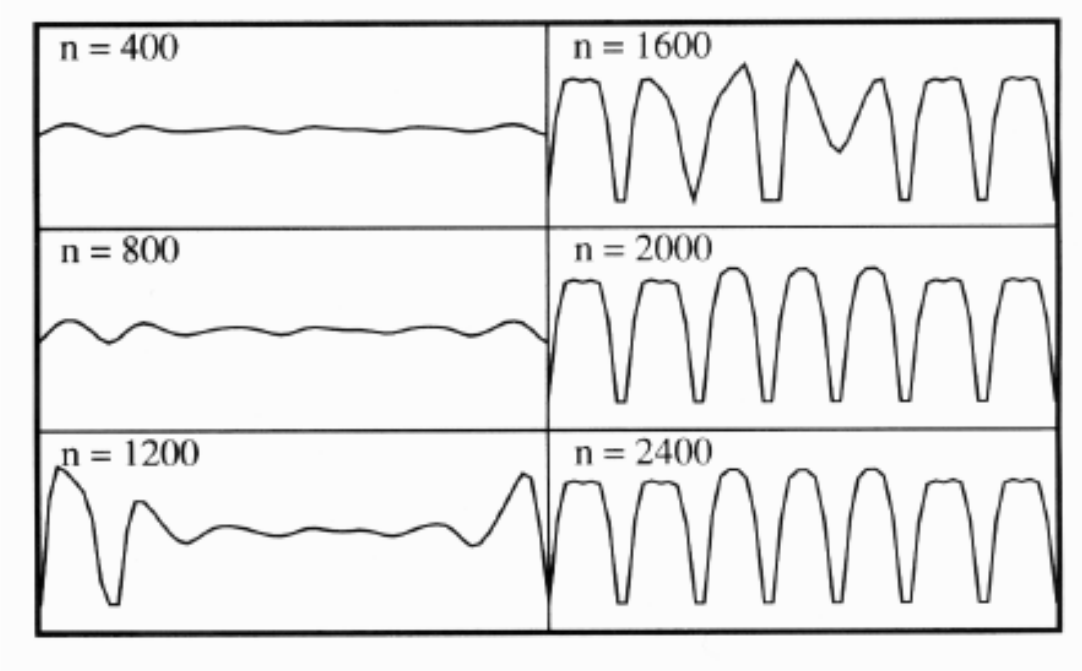
Turing's Solution

$$\Delta a_i = D_a(a_{i+1} + a_{i-1} - 2a_i) + k(16 - a_i b_i)$$

$$\Delta b_i = D_b(b_{i+1} + b_{i-1} - 2b_i) + k(a_i b_i - b_i - 12 - \beta_i)$$

- a_i : concentration of 1st morphogen at i^{th} cell. (inhibitor)
- b_i : concentration of 2nd morphogen at i^{th} cell. (activator)
- D_a : diffusion rate of a.
- D_b : diffusion rate of b.
- β : random substrate
- k : reaction rate
- Initial concentrations of a, b: 4

1-D Simulation

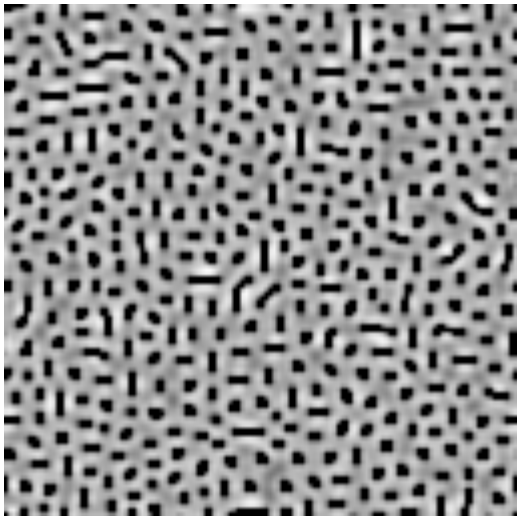


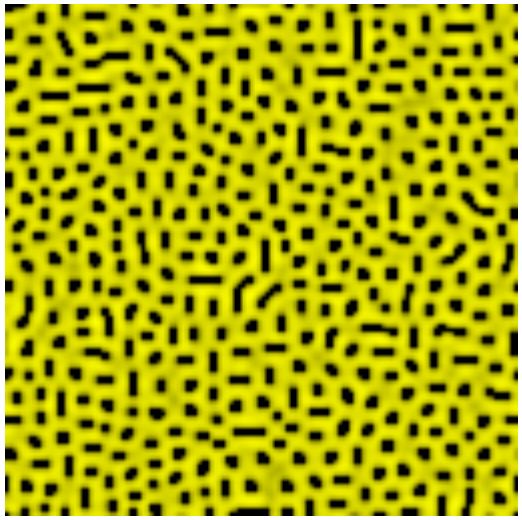
Concentration of b over time.

2-D Simulation

$$\Delta a_{i,j} = D_a(a_{i+1,j} + a_{i-1,j} + a_{i,j+1} + a_{i,j-1} - 4a_{i,j}) + k(16 - a_{i,j}b_{i,j})$$

$$\Delta b_{i,j} = D_b(b_{i+1,j} + b_{i-1,j} + b_{i,j+1} + b_{i,j-1} - 4b_{i,j}) + k(a_{i,j}b_{i,j} - b_{i,j} - 12 - \beta_{i,j})$$





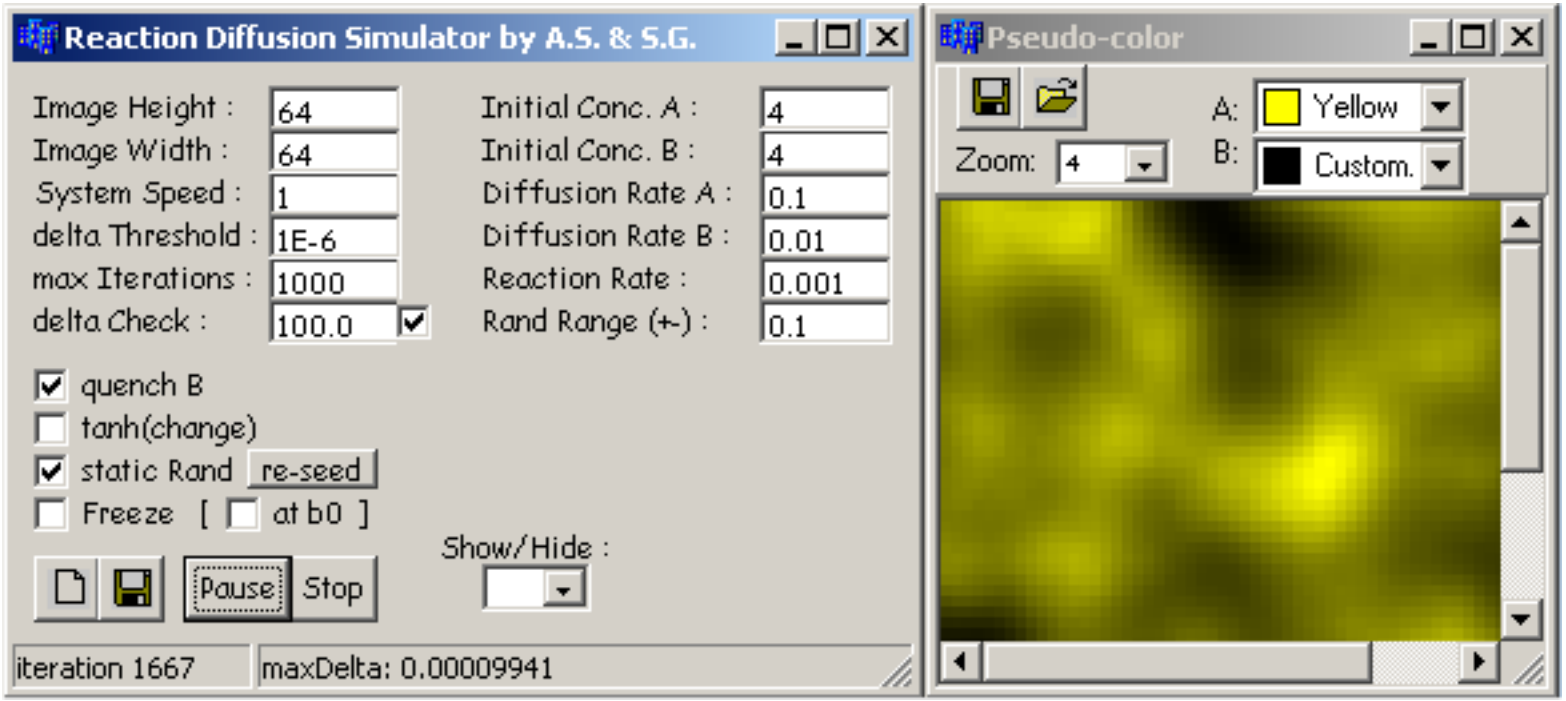
a b [a: black, b: yellow]

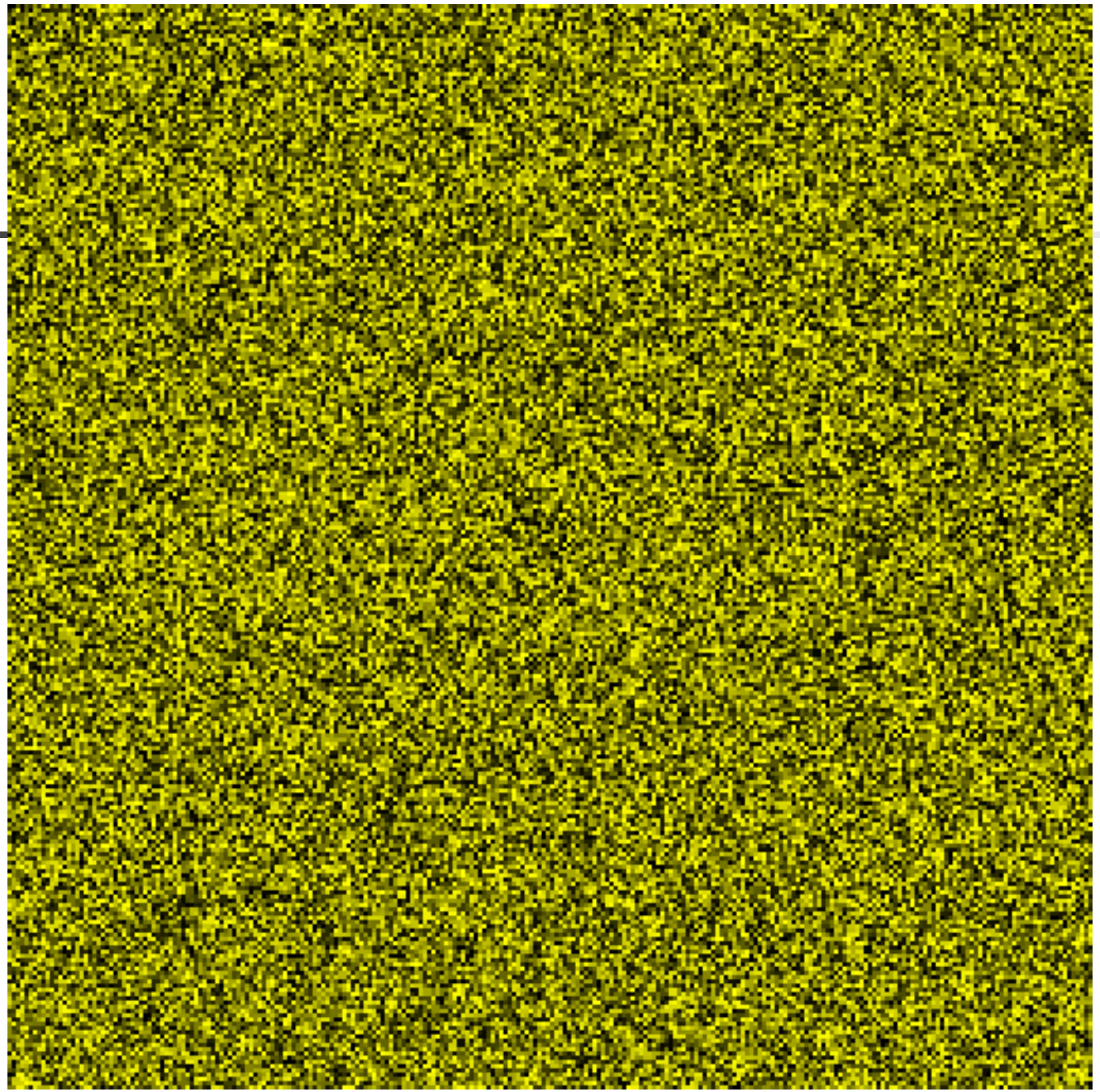
D_a=0.1 D_b=0.02 β=0.1 k=0.02

Reaction Diffusion Simulator

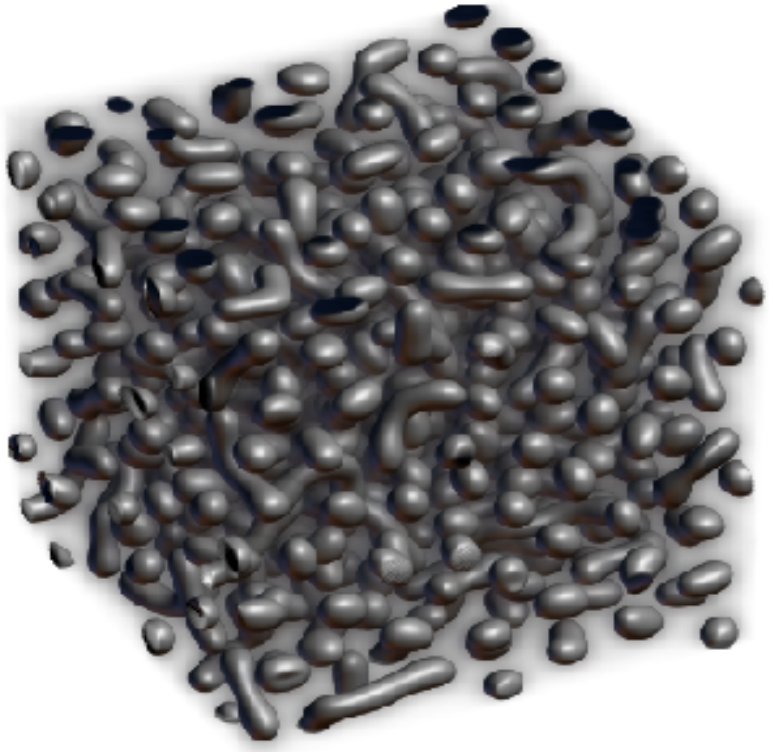
- Available at:

  http://menekse.ceng.metu.edu.tr/Ceng/566/RD/



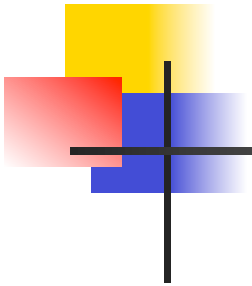


3-D Simulation



$D_a=0.125\ D_b=0.03125\ \beta=0.1\ k=0.0125$

Possible Trends



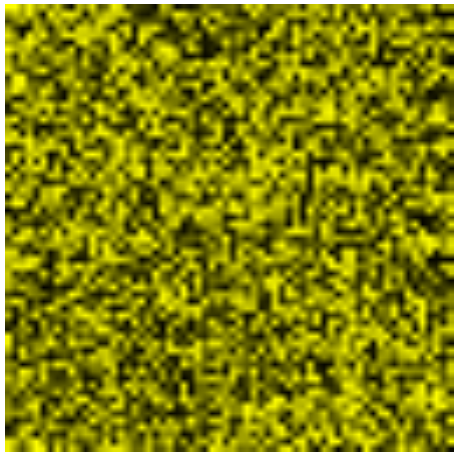
- Oscillating chemical concentrations
- Unbounded increase (decrease)
- **A Steady State**
  - Different diffusion rates
  - Random perturbation

No Reaction Case

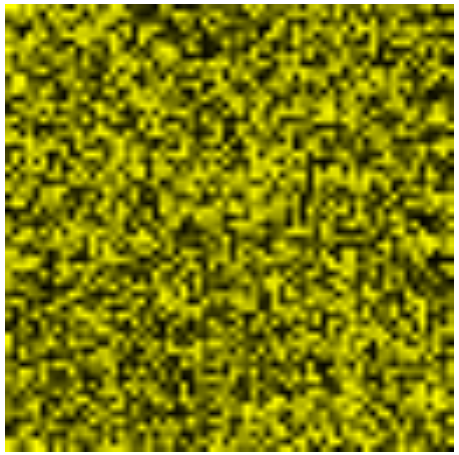
a b

D_a=0.1 D_b=0.02 β=0.1 **k=0.0**

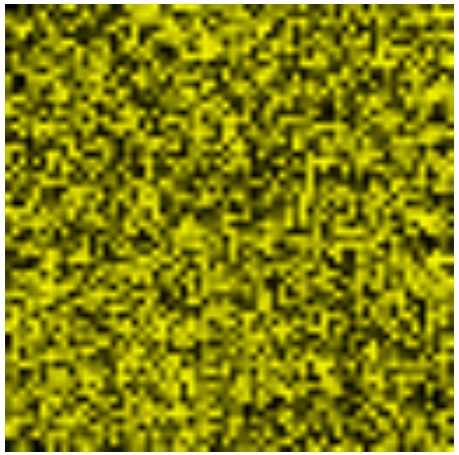
No Diffusion Case



n=1



n=1000

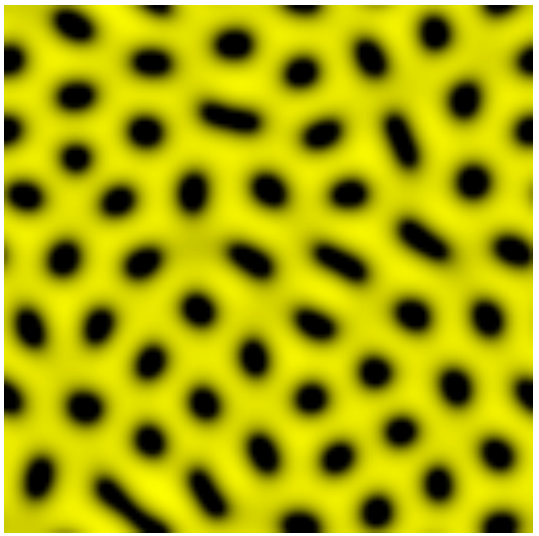


n=2000

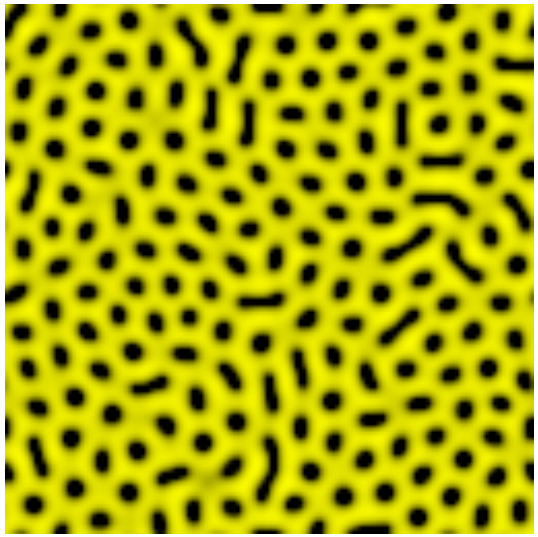
$$D_a=0.0\ D_b=0.0\ \beta=0.1\ k=0.01$$

Parameter-Game: k

k=0.001



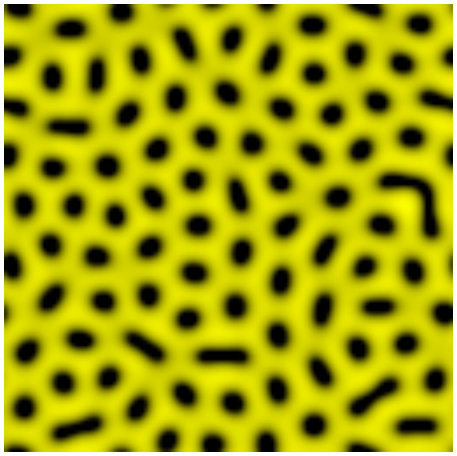
k=0.005



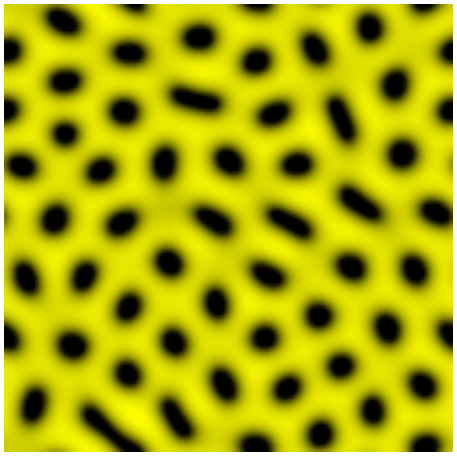
k=0.01

D_a=0.1 D_b=0.02 β=0.1

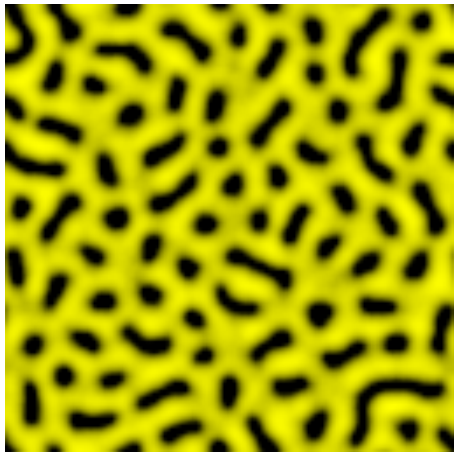
Parameter: β



β=0.05

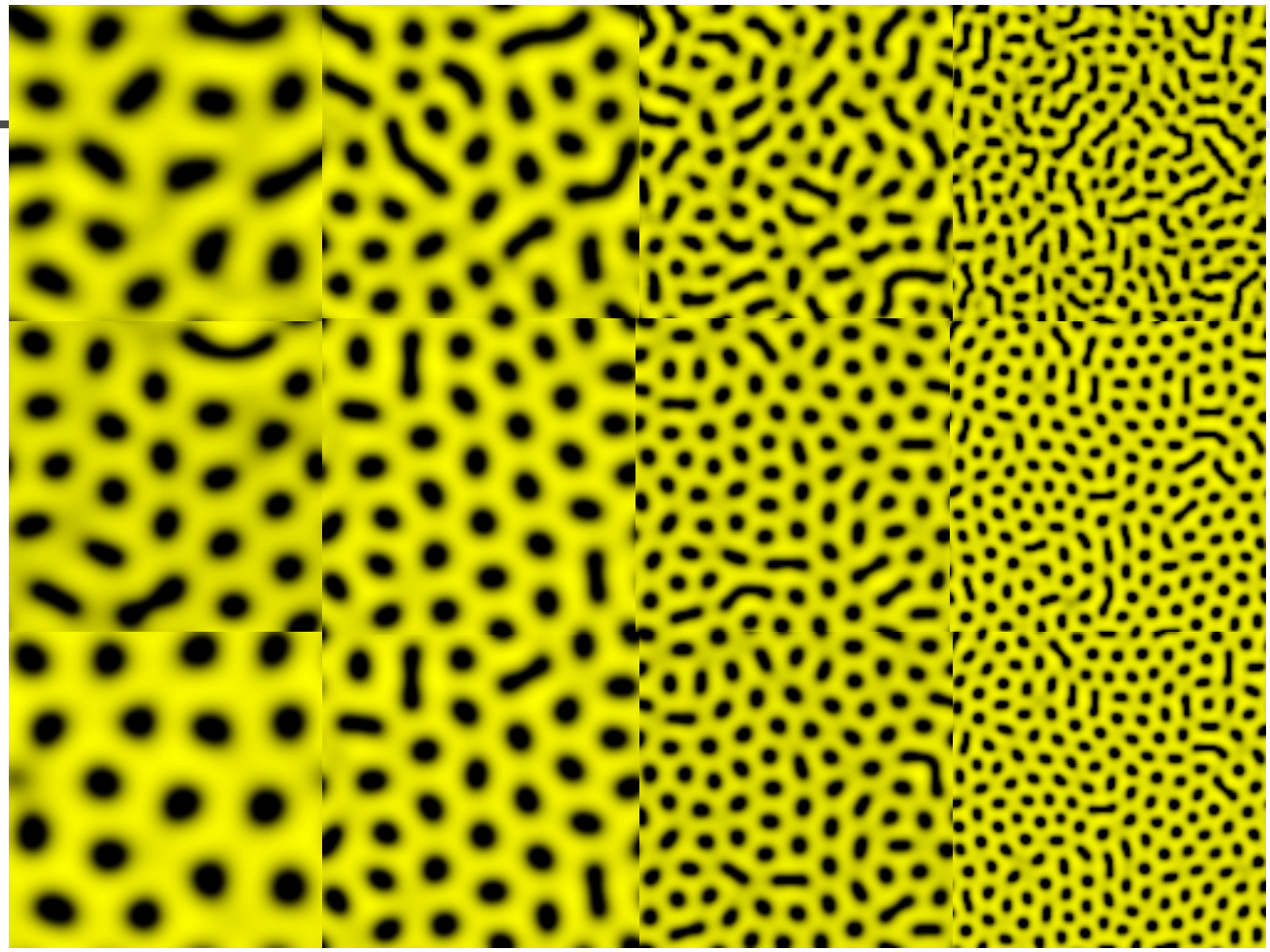


β=0.1



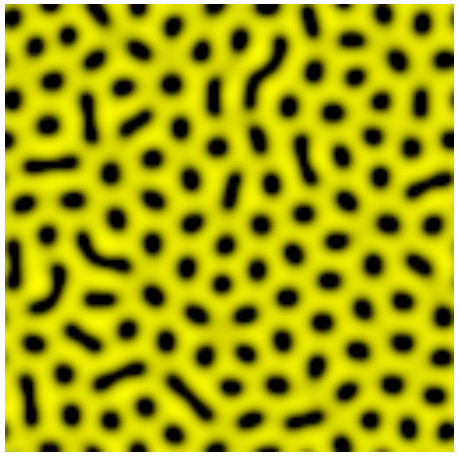
β=3

D_a=0.1 D_b=0.02 k=0.005

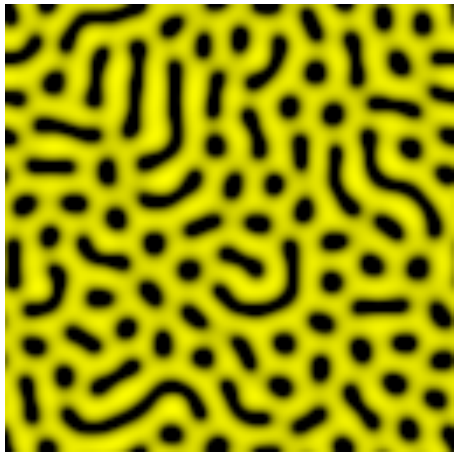


$D_a=0.1\ D_b=0.02$

Parameter: D_a / D_b



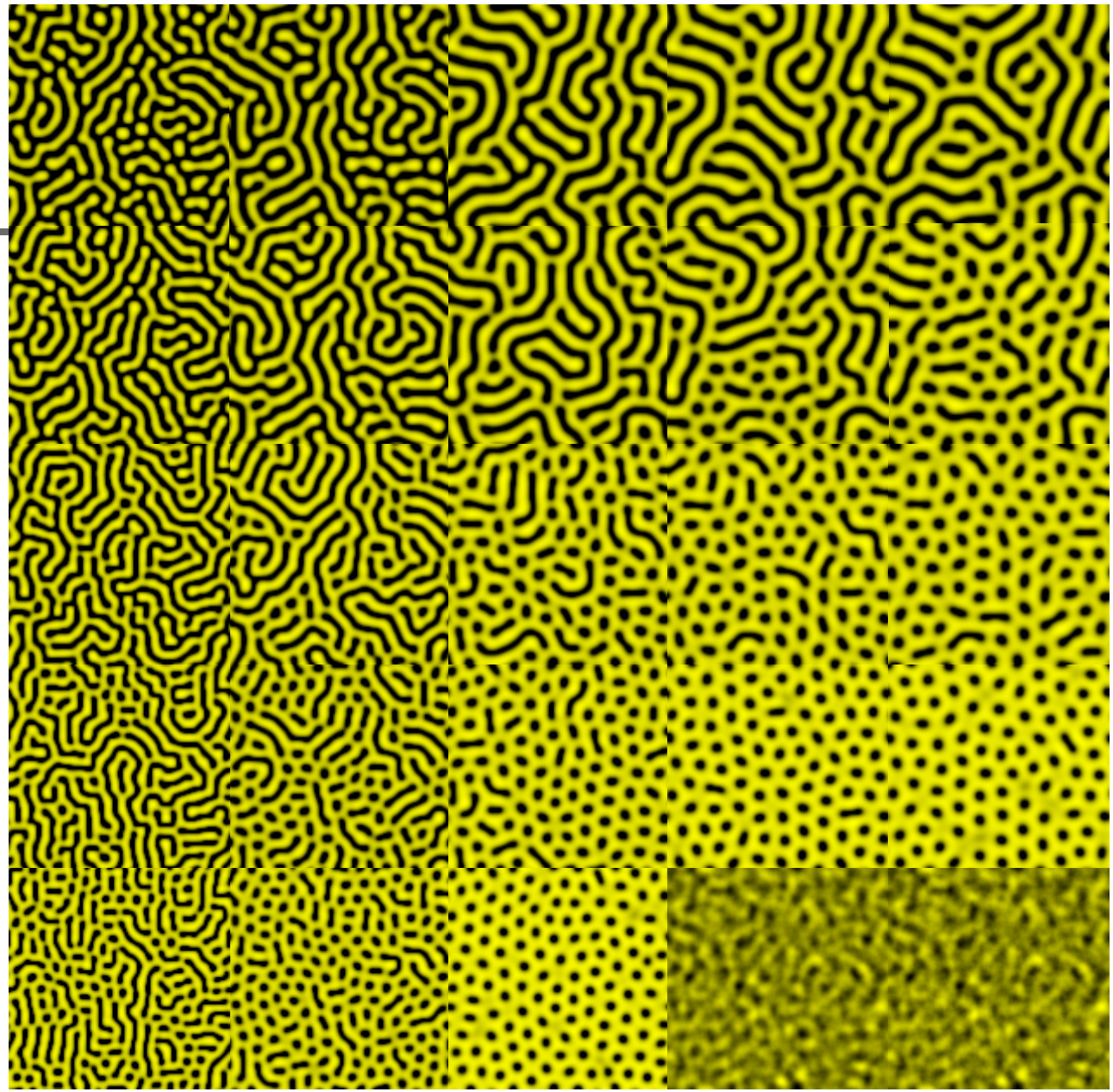
D_a=0.08

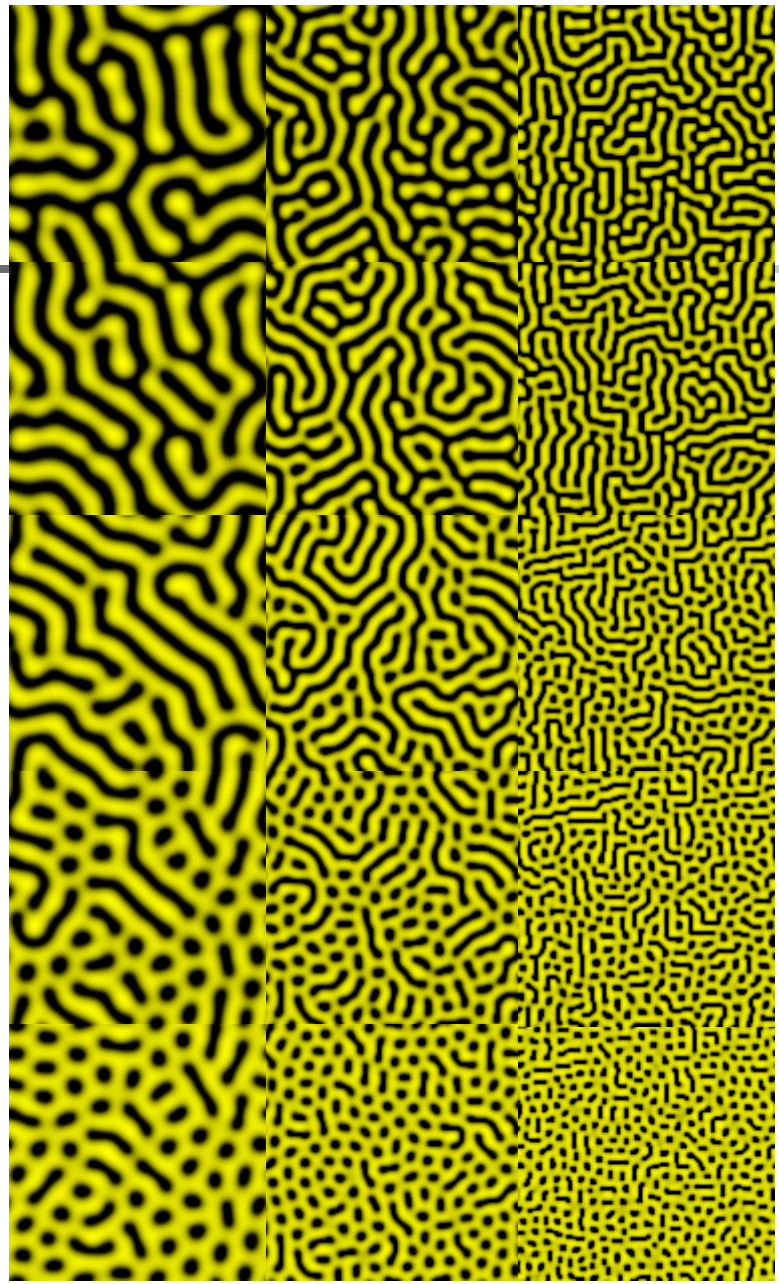


Da=0.1

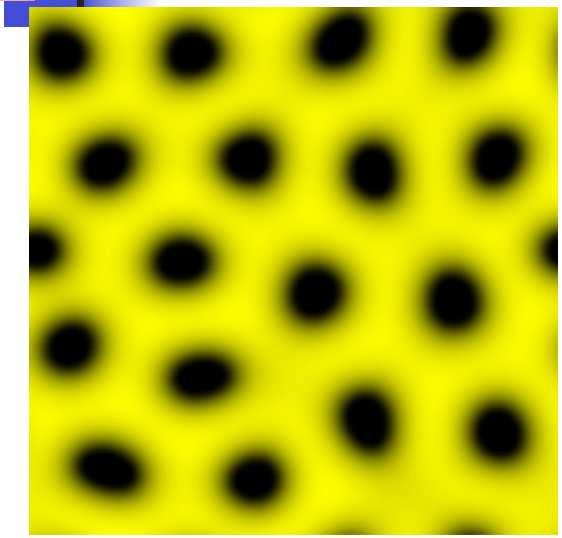
Da=0.2

D_b=0.02 β=0.1 k=0.005

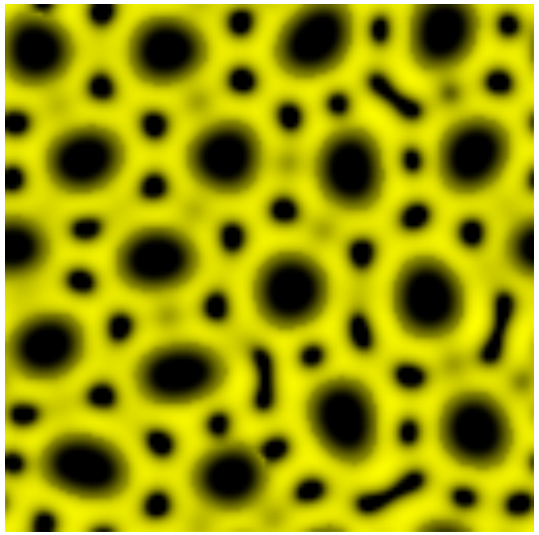

Da=0.2
Da=0.15
Da=0.1
Da=0.08
Da=0.06
β=0.1
k=0.005
D_b=0.007
D_b=0.01
D_b=0.015
D_b=0.018
D_b=0.02


Da=0.2
Da=0.15
Da=0.1
Da=0.08
Da=0.06
k=0.002
k=0.005
k=0.01
β=0.1
D_b=0.01

Cascading



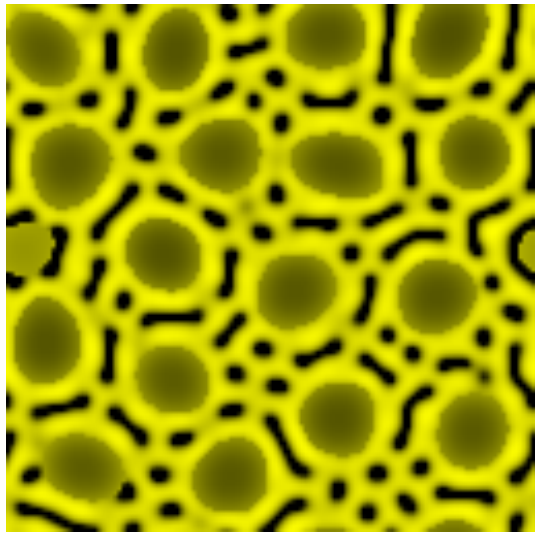
k=0.001, n=30000



Freeze b:[0-4]

k=0.01

Cheetah



Freeze b:[0-4]→4

k=0.01

Leopard

D_a=0.1 D_b=0.02 β=0.05

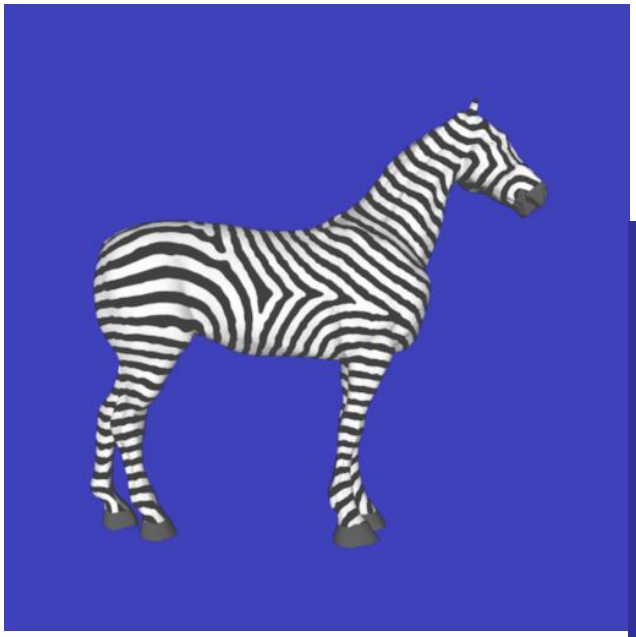
History of RD

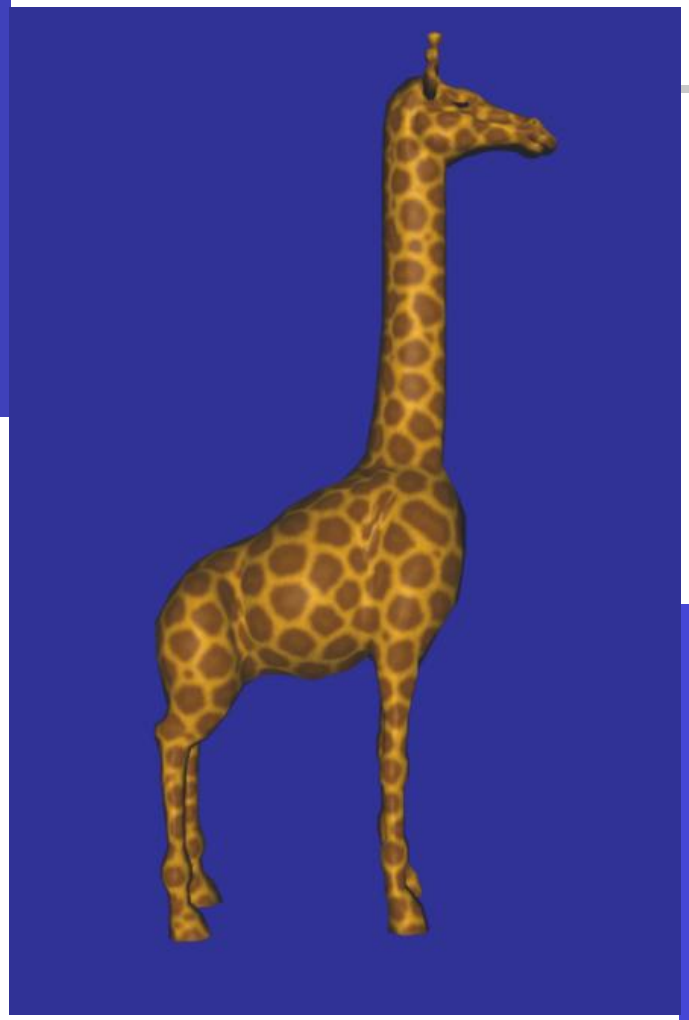
- Turing (1952)
  - RD system on a sphere may be responsible for triggering gastrulation in the embryo.
- Bard and Lauder (1974)
  - Computer simulations → Patterns generated by RD not regular enough to explain patterns in development.
  - Can explain less regular patterns: leaf organization, distribution of hair follicles.

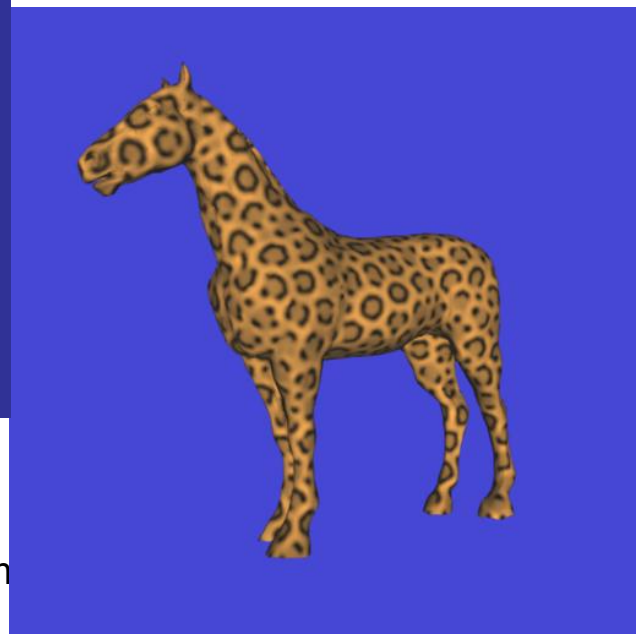
- Bard (1981), Murray (1981) independently
  - RD can explain the patterns on coats of animals.
- Bard (1981)
  - Spot and stripe patterns.
  - Small, white spots on a deer.
  - Large, dark spots on a giraffe.
- Murray (1981)
  - Spot-size dependent on size of animal.
  - Paterns found on butterfly wings.

- Meinhardt (1982)
  - Stripe patterns (by 5-morphogen RD)
  - Veins on a leaf.
- Swindale (1980)
  - Simulation by activation/inhibition between synapses.
- Young (1984)
  - Irregular striped patterns
  - Ocular dominance columns in mammalian visual system.
- Meinhardt and Klinger (1987)
  - Patterns of pigment found on mollusc shells

- Kauffman et al. (1978), Lacalli (1990), Hunding et al. (1990)
  - Segmentation of fruit fly (Drosophila) embryos
- Turk (1991)
  - Cascading
  - Clusters of spots on leopards and jaguars (rosettes)
  - Zebra's pajamas.
  - Mapping on arbitrary surfaces.







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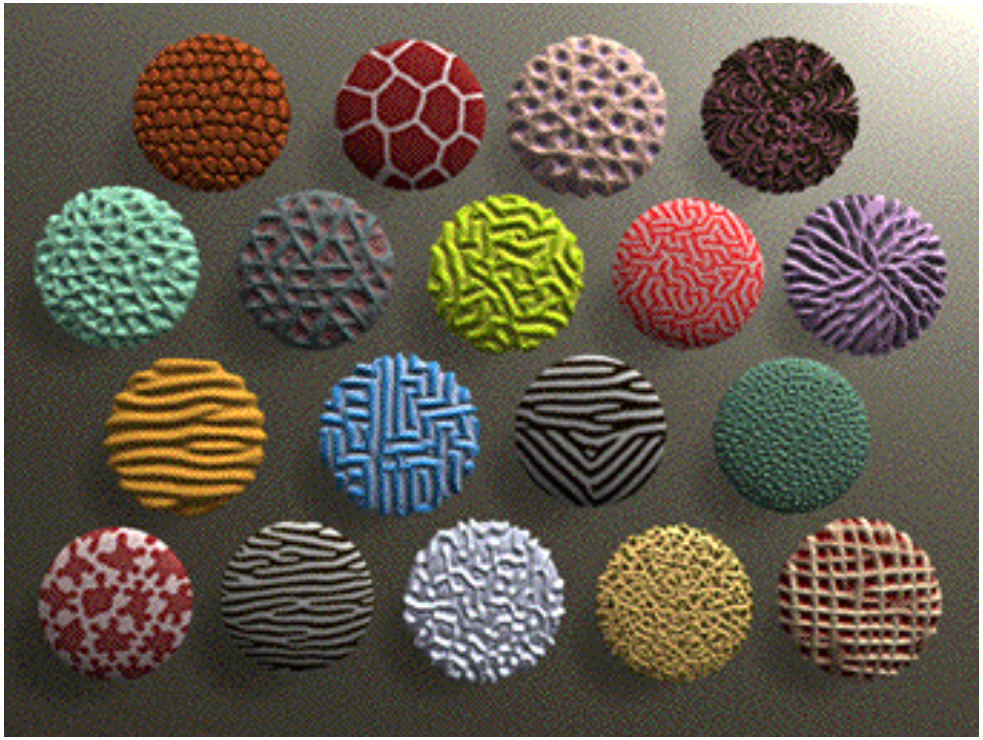
- Whitkin and Kass (1991).
  - Emphasize anisotropy.
  - “diffusion map”: diffusion varies across a surface.



Space Cookie

Pearson (1993)

- Gray-Scott Model
- Well-Defined range of behavior for parameters.

$$U + 2V \rightharpoonup 3V$$

$$V \rightharpoonup P$$

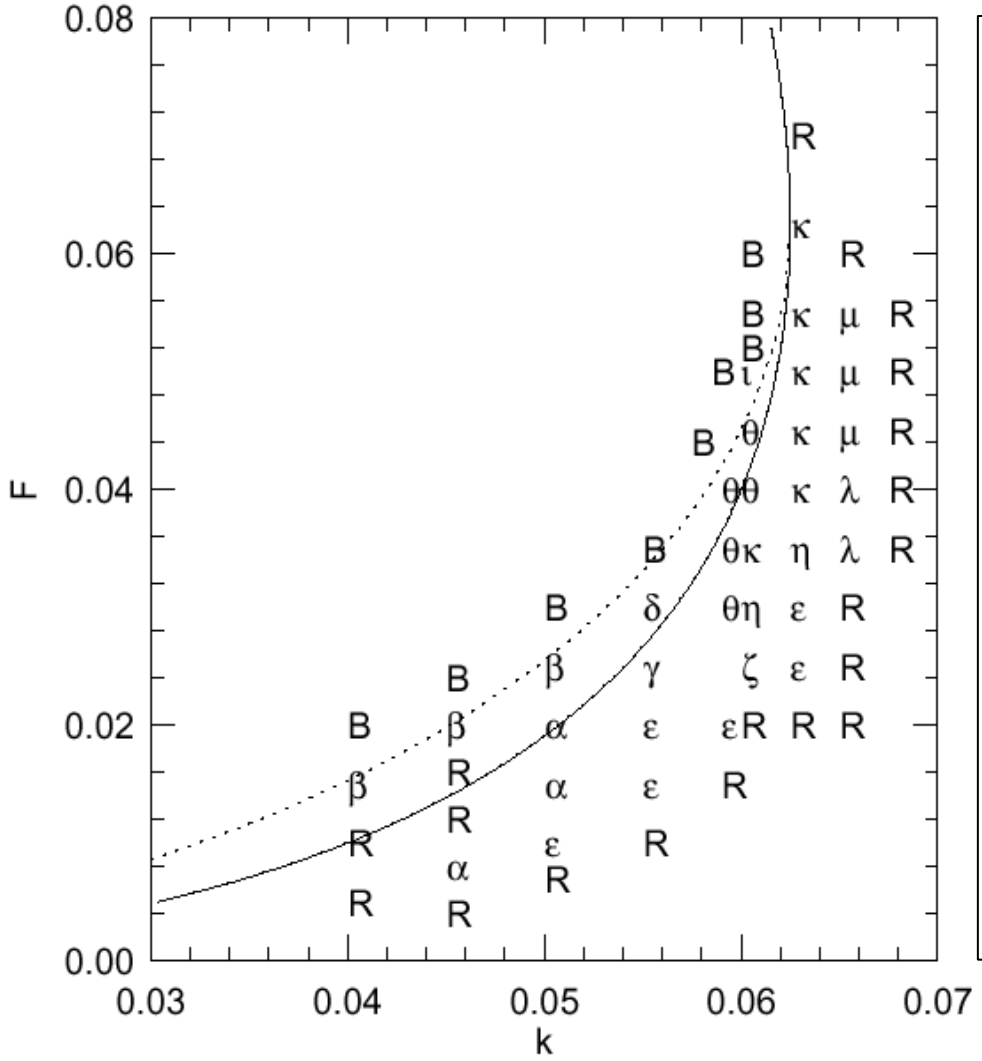
$$\frac{\partial U}{\partial t} = D_u \nabla^2 U - UV^2 + F(1-U)$$

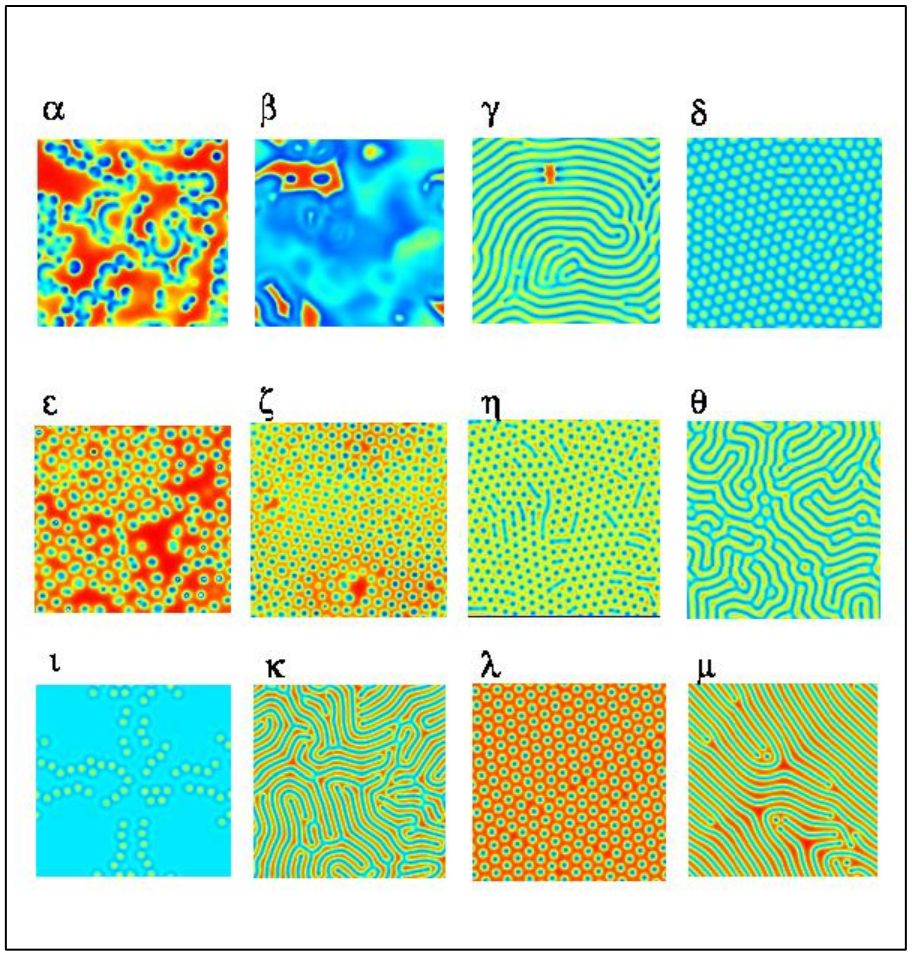
$$\frac{\partial V}{\partial t} = D_v \nabla^2 V + UV^2 - (F+k)V$$

D_u = 2E-5 and D_v = 1E-5

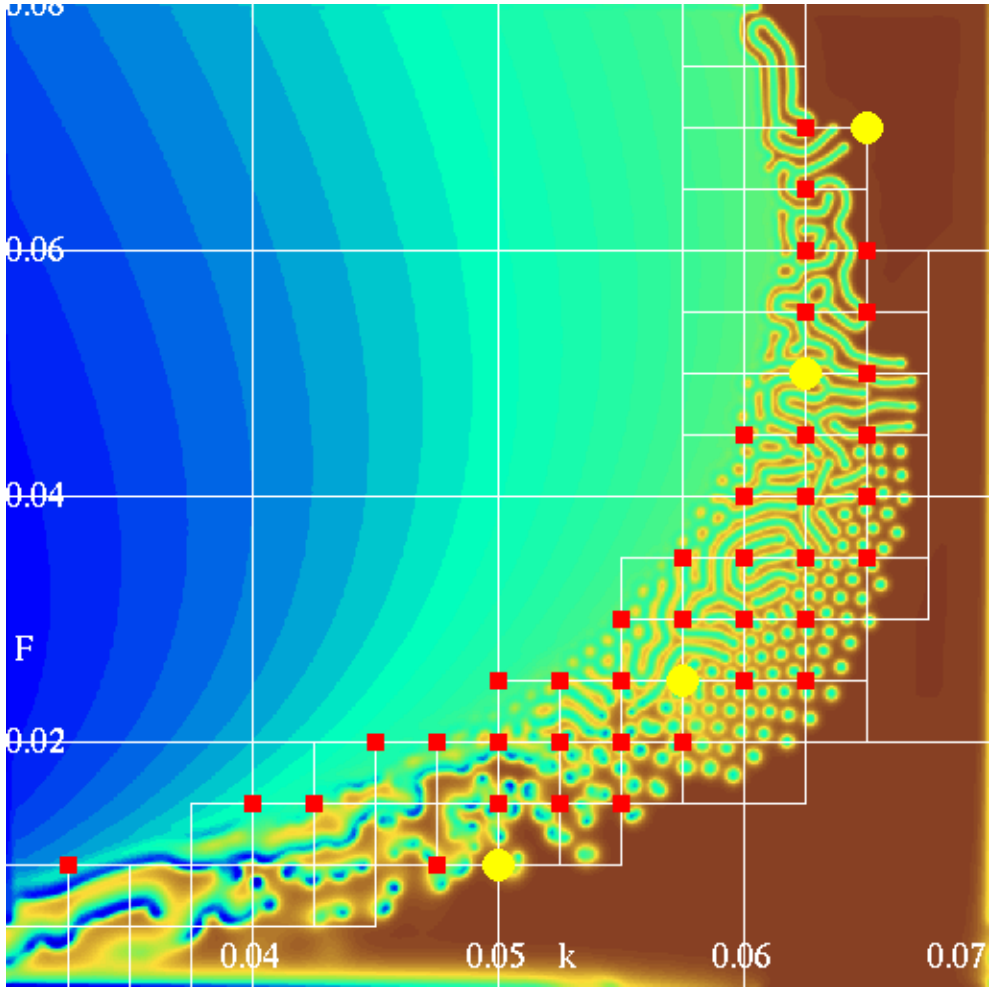
F: rate of the process that feeds U and drains U,V and P

k: rate of conversion of V to P

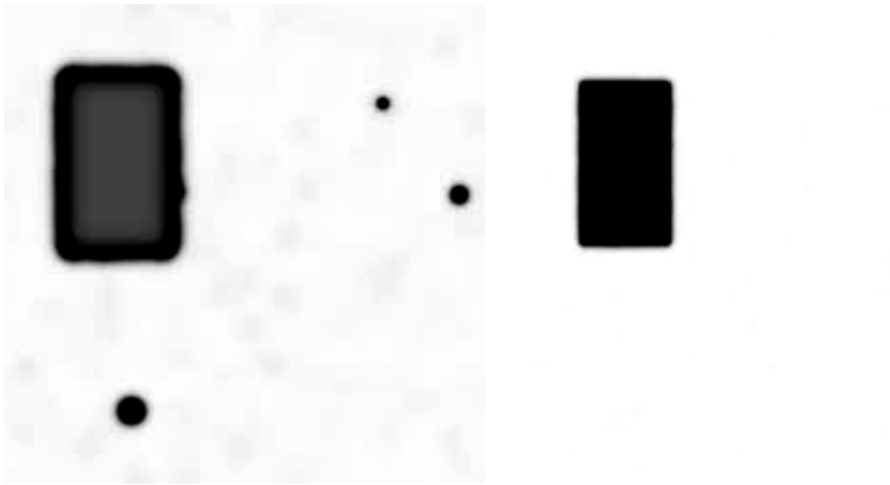

0.08
0.06
0.04
0.02
0.00
F
0.03
0.04
0.05
0.06
0.07
k


α
β
γ
δ
ε
ζ
η
θ
ι
κ
λ
μ

Xmorphia

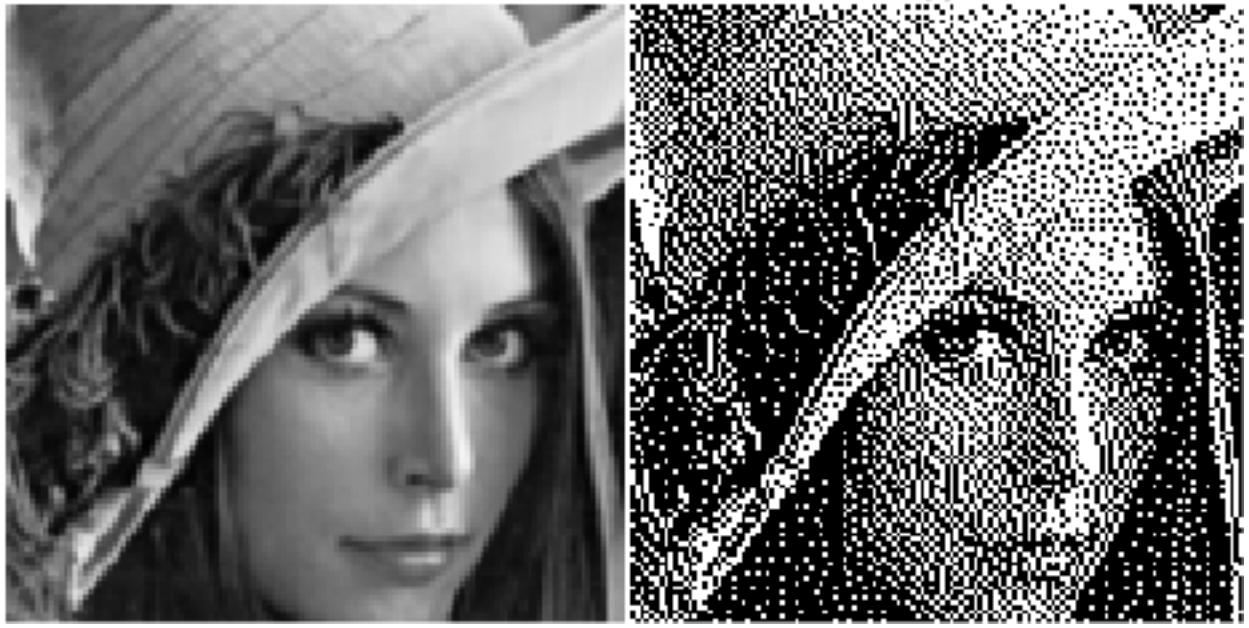


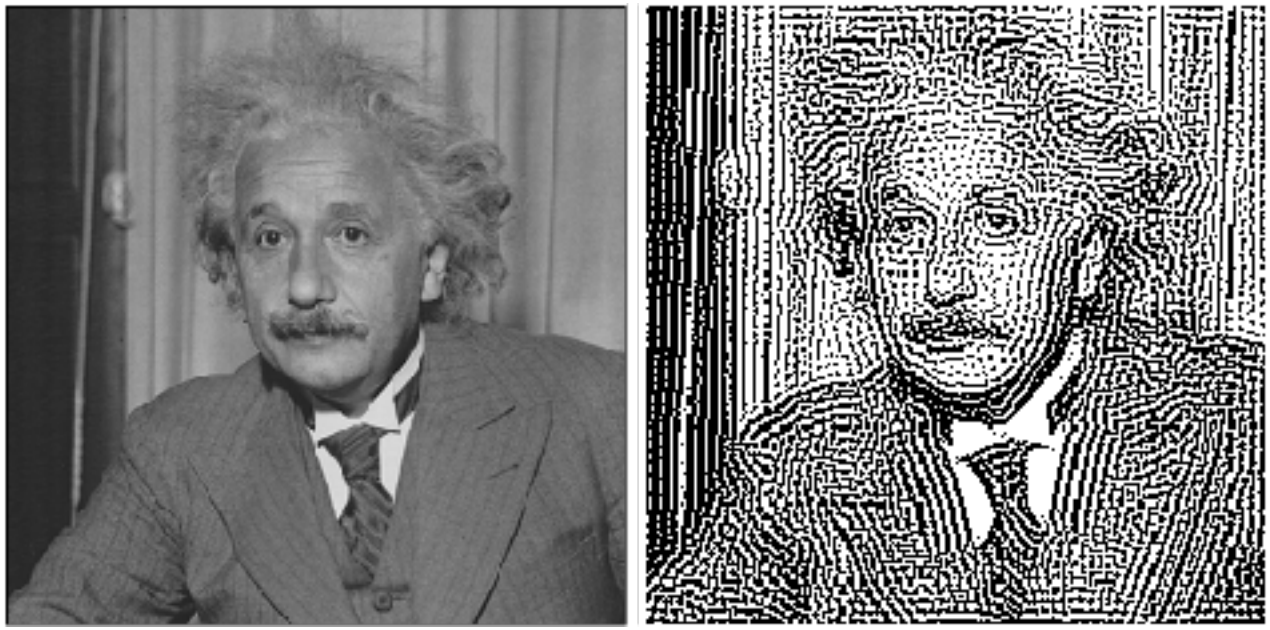
Xmorphia



M-Lattice

- Sherstinsky, Picard (1994)
  - State variables are guaranteed to be bound.
  - Applied to image-restoration and half-toning.





(a)

(b)

Figure 1: Orientation-sensitive color halftoning. (a) the original "Marty" image; (b) the "Marty" image adaptively halftoned using orientation information at each pixel of the original.

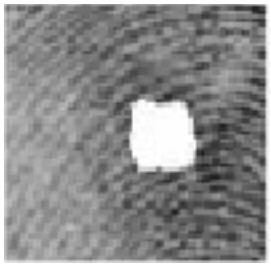
(a)

(b)

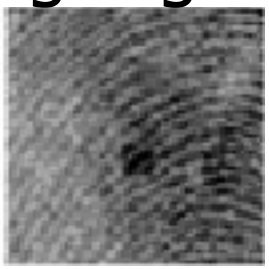
Figure 2: Orientation-sensitive color halftoning. (a) the original "Betty" image; (b) the "Betty" image adaptively halftoned using orientation information at each pixel of the original.

Texture Completion

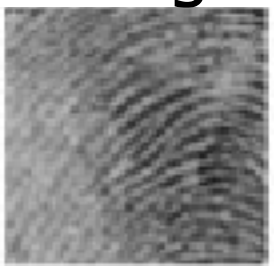
- Acton, Mukherjee, Havlicek, Bovik (2001).
  - Reconstruction of large missing regions of homogeneous oriented textures.
  - RD seeded with noise identically distributed to surrounding region to match graylevel ution.



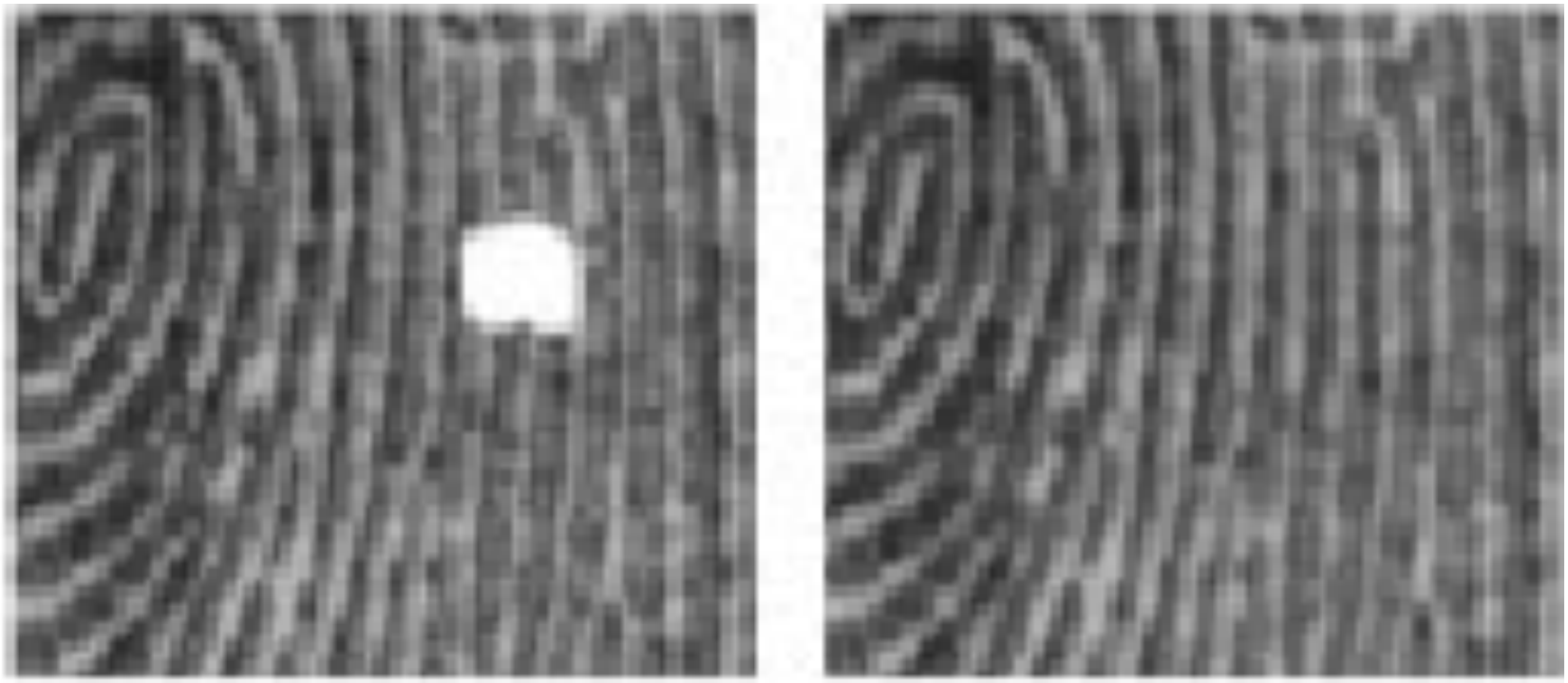
occluded

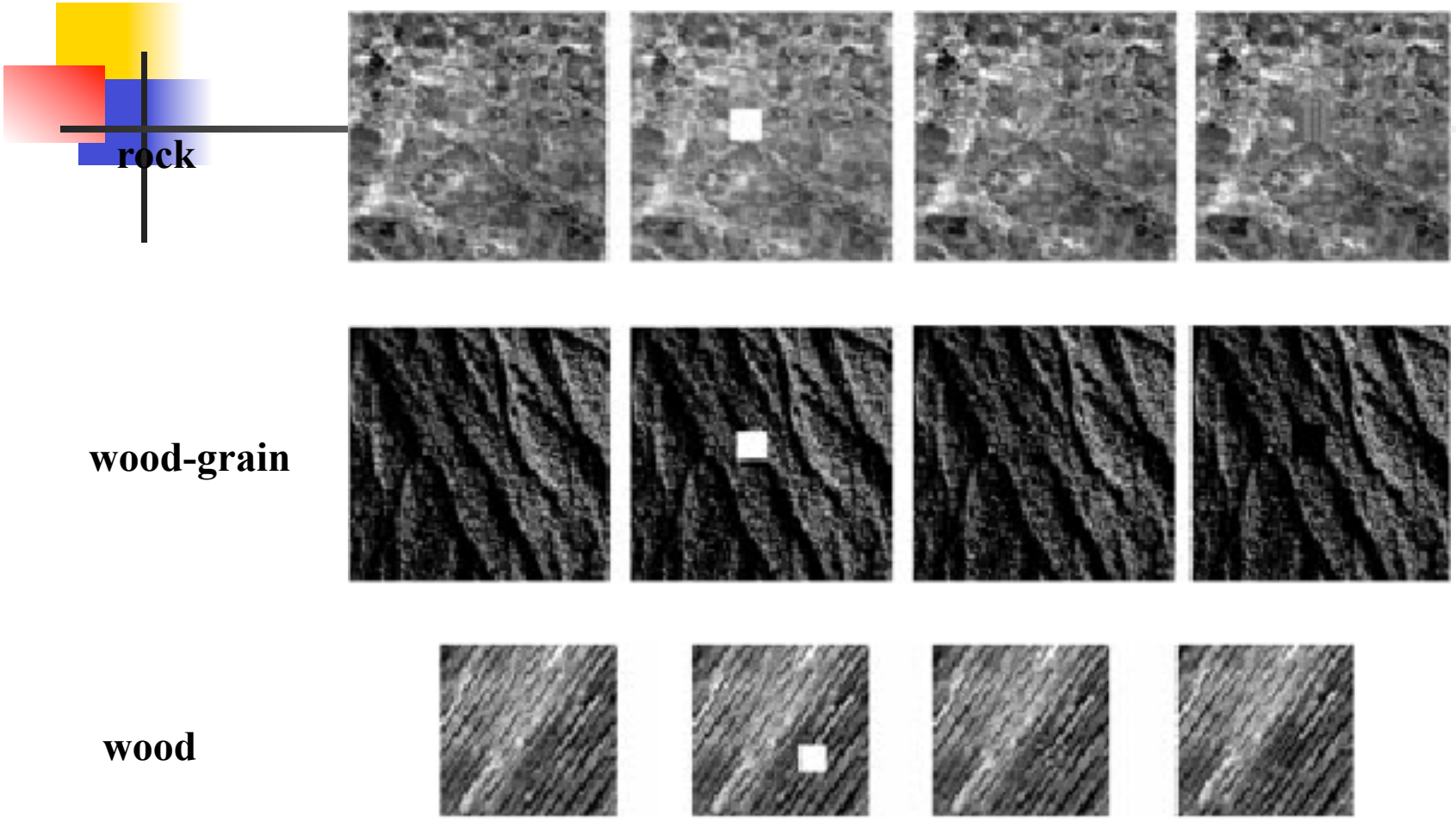


stripe formation



AM-FM RD




rock
wood-grain
wood
AM-FM RD
Level-line method

Web-Resources

- Code Zebra (collection of RD links)

  http://www.codezebra.net/zebraSite/archive.html

- Greg Turk's page:

  www.gvu.gatech.edu/people/faculty/greg.turk/reaction_diffusion/reaction_diffusion.html

- Xmorphia:

  http://www.cacr.caltech.edu/ismap/image.html

- 3D images:

  http://www.cs.utah.edu/~gk/papers/tvcg00/node7.html

- Visual models of morphogenesis

  http://www.cpsc.ucalgary.ca/Research/bmv/

References

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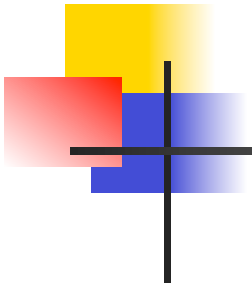

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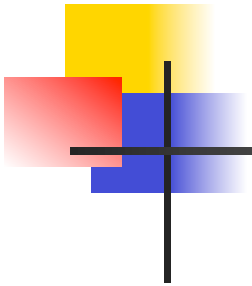

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