

THE COMPLEXITY OF EXTENDING A GRAPH IMBEDDING

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Abstract

Given a graph G and an imbedding I_0 of a subgraph G_0 of G into a 2-manifold M , the imbedding extension problem is to determine if there exists an imbedding I of G into M extending I_0 . We show that this problem is complete in nondeterministic polynomial time, even if

- (1) G is cubic (the maximum valence of any vertex of G is 3), or
- (2) I_0 is quasiplanar (each closed face of I_0 is homeomorphic to a disk).

This complexity result is tight in the sense that if G is cubic and I_0 is quasiplanar, then the imbedding extension problem can be solved in quadratic time.

We also show that the problem of counting nonisotopic imbeddings of a cubic graph extending a given quasiplanar imbedding is a complete counting problem.

1. Introduction

Considerable research effort [Lempel, Even, and Cederbaum, 1967; Hopcroft and Tarjan, 1974] has been devoted to efficient algorithms for recognition of planar graphs: graphs which have an imbedding into the plane. In general, the imbedding problem for graph G and topological surface M is to determine if G has an imbedding into M . We have recently [Reif, 1978] developed a polynomial-time algorithm for the imbedding problem for any fixed orientable surface.

Our algorithm for the imbedding problem and many previous algorithms (for example, the [Hopcroft and Tarjan, 1974] planar graph recognition algorithm and the [Filotti, 1978a and 1978b] cubic, toroidal graph recognition algorithm) have the following general form:

- (1) Initially we construct a fixed number of imbeddings of subgraphs of graph G into surface M .
- (2) Next, we attempt to extend each of these partial imbeddings to an imbedding of G into M .

This paper is concerned with the time complexity of the imbedding extension problem: given a graph G and an imbedding I_0 of a subgraph of G into surface M , determine if G has an imbedding into M extending I_0 . We have shown [Reif, 1978] that if the surface M is fixed and orientable, then the imbedding extension problem is in polynomial time. Unfortunately, we show here that the imbedding extension problem for arbitrary surfaces is NP-complete in the sense of [Cook, 1971]. This implies that the imbedding problem for graph G and arbitrary surface M cannot be solved efficiently by an algorithm of the above form, unless $P = NP$.

In the next section we give some preliminary definitions on computational complexity, surfaces, graphs and their imbeddings into surfaces. Section 3 presents our proof that the imbedding extension problem is NP-complete. Section 4 gives a polynomial-time algorithm for solving a restricted class of imbedding extension problems. Section 5 concludes the paper with some related open problems.

2. Preliminary Definitions

2.1 Computational Complexity

We consider in this paper the complexity of various imbedding problems. As usual, we view these problems as language recognition problems, say over a fixed alphabet Σ . Let $P(NP)$ be the class of languages recognized by (non)deterministic Turing Machines in polynomial time. Let NL be the class of languages recognized by nondeterministic Turing Machines in log-space. A counting Turing Machine is a nondeterministic Turing Machine augmented with an oracle for counting all accepting instances. Let $\#P$ be the class of languages recognized by counting Turing Machines.

Let L, L' be languages over alphabet Σ . A polynomial-time (log-space) reduction from L' to L is a polynomial-time (log-space) computable function $f: \Sigma^* \rightarrow \Sigma^*$ such that for each $x \in \Sigma^*$, $x \in L'$ if and only if $f(x) \in L$. The reduction is parsimonious if for each $y \in L$, there is a $x \in L'$ such that $f(x) = y$. Let \mathcal{L} be a class of languages over Σ . L is \mathcal{L} -hard if there is a log-space reduction from each $L' \in \mathcal{L}$ to L , and L is \mathcal{L} -complete if in addition, $L \in \mathcal{L}$.

Let k -CNF be the set of boolean formulas in conjunctive normal form with k literals per clause. Let k -(UN)SAT be the problem of testing if a formula of k -SAT is satisfiable. The following will be of use:

Proposition 2.1: 2-SAT is linear-time.

Proposition 2.2: 2-UNSAT is NL-complete [Jones, Lien, and Lauser, 1976].

Proposition 2.3: 3-SAT is NP-complete [Cook, 1971].

Let #k-SAT be the problem of counting the satisfying instances of a formula in k-SAT.

Proposition 2.4: #2-SAT is #P-complete [Valiant, 1977].

2.2 Graphs

A graph $G = (V, E)$ consists of a set of vertices V and a set of edges E consisting of unordered pairs of distinct vertices $\{u, v\} \subseteq V$. The size of G is $|V| + |E|$. The valence of G is the maximum number of edges containing a given vertex. A graph of valence 3 is cubic. A path is a sequence of edges $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{k-1}, v_k\}$ and is a cycle if $v_1 = v_k$. Given graphs $G = (V, E)$ and $G' = (V', E')$, let $G + G'$ be the graph $(V \cup V', E \cup E')$. Let G' be a subgraph of G ($G' \subseteq G$) if $V' \subseteq V$ and $E' \subseteq E$.

A graph $G = (V, E)$ will frequently be identified with a 1-simplicial complex:

each vertex $v \in V$ is considered a point in Euclidean 3-space and each edge $\{u, v\} \in E$ is considered an arc from u to v . A pair of edges $\{u, v\}, \{u', v'\} \in E$ may intersect only at their endpoints $u, v, u',$ or v' .

2.3 Topological Surfaces

A 2-manifold M is a connected, topological space in which the neighborhood of each point is homeomorphic to an open disk. M is closed if the boundary of M is M itself. A surface will be assumed to be a closed 2-manifold.

A surface M is orientable if the points in the neighborhood of each curve C on M may be consistently oriented to either the right or left of C (else M is nonorientable). In general, any orientable surface M may be characterized (up to homeomorphism) as a sphere with an addition of $\gamma \geq 0$ "handles." The integer γ is the genus of M .

For example, the torus is of genus 1. Similarly, any nonorientable surface M may be characterized (up to homeomorphism) as a sphere with $\kappa > 0$ "crosscaps." For example, the projective plane has exactly one crosscap. These characterizations of surfaces are due to [Fréchet and Fan, 1967].

2.4 Graph Imbeddings

Let $I: G \rightarrow M$ be a homeomorphism of graph G into surface M . Each maximal, connected region F of $I(G) - M$ is an open face of I . The boundary of F , $B(F)$, is considered to be oriented in a clockwise fashion and consists of a cycle of G . The closed face \bar{F} consists of F plus its boundary $B(F)$. An (2-cell) imbedding is a homeomorphism $I: G \rightarrow M$ in which each open face is homeomorphic to an open disk. An imbedding I is quasiplanar if each closed face is homeomorphic to a disk. Any imbedding $I: G \rightarrow M$ with f faces satisfies Euler's formula:

$$|E| - |V| + f = 2 - 2\gamma \text{ if } M \text{ is an orientable surface of genus } \gamma, \\ = 2 - \kappa \text{ if } M \text{ is a nonorientable surface with} \\ \kappa \text{ crosscaps.}$$

2.5 Combinatorial Imbeddings

We describe here an elegant combinatorial characterization of graph imbeddings given by [Edmonds, 1960, and Youngs, 1963].

Let a combinatorial imbedding of graph G be a set of cycles CI of G such that each edge of G occurs exactly twice in CI . An imbedding I of G into surface M is represented by the combinatorial imbedding $\{B(F) \mid F \text{ is a face of } I\}$. A combinatorial imbedding CI represents orientable imbeddings just in the case each edge of G is traversed in opposite directions in its two appearances in CI .

Given a combinatorial imbedding CI of G , we construct a surface M by associating a disk D_B with boundary B for each $B \in CI$; then "glueing" these disks together by identifying pairs of boundary segments associated with a unique edge of G . The identity mapping I from G into the resulting surface M is represented by the original combinatorial imbedding CI . Note that any pair of isotopic imbeddings of G into M are represented by a unique combinatorial imbedding.

Thus, to determine if graph G has an imbedding into surface M , we can enumerate all combinatorial imbeddings of G and test (by applying Euler's formula) if any represents an imbedding of G into M . Consequently, we have:

Theorem 2.1: The imbedding problem is in NP.

Given a combinatorial imbedding CI of graph G and combinatorial imbedding CI_0 of some subgraph $G_0 \subseteq G$, CI is consistent with CI_0 if CI_0 may be derived from CI by repeatedly merging pairs of cycles and deleting their oppositely directed common subsequences. Note that if imbeddings $I : G \rightarrow M$, $I_0 : G_0 \rightarrow M$ are represented by IC , IC_0 , then I extends I_0 if and only if CI is consistent with CI_0 .

Consider the imbedding extension problem for graph G and partial imbedding $I_0 : G_0 \rightarrow M$ represented by combinatorial imbedding CI_0 . To determine if there is an imbedding $I : G \rightarrow M$ extending I_0 , we can enumerate all combinatorial imbeddings consistent with CI_0 and test if any represents an imbedding $I : G \rightarrow M$. We have shown:

Theorem 2.2: The imbedding extension problem is in NP.

3. The Imbedding Extension Problem is NP-complete

We show that the imbedding extension problem is NP-hard by a polynomial-time reduction from SAT: satisfiability of Boolean formulas in CNF form. Let $S = \{X_1, \dots, X_S\}$ be a set of boolean variables, taking values in $\{0,1\}$. Let \bar{X}_i be the negation of boolean variable $X_i \in S$ and let $\bar{\bar{X}}_i = X_i$. Let a literal ℓ be a boolean variable or its negation. Let a clause C be a list of literals, and let a boolean formula be a set Q of clauses.

Let L be the set of literals occurring in BF , and their negations. Given truth assignment $\tau: S \rightarrow \{0,1\}$, let $\tau(Q) = 1$ iff for each $C \in Q$ there exists some literal $\ell \in C$ such that $\tau(C) = 1$. Q is satisfiable if there exists a truth assignment τ with $\tau(Q) = 1$.

We now define a partially imbedded graph G and will show that the imbedding of G is extendable if and only if Q is satisfiable.

Associated with each clause $C \in Q$, let there be distinguished vertices u_C, v_C . For each literal $\ell \in L$, let there be a distinguished vertex w_ℓ . Let $C(\ell) = \{C_1, \dots, C_k\}$ be those clauses of Q containing literal ℓ . Let B_ℓ be the simple cycle containing distinguished vertices in order $w_\ell, u_{C_1}, \dots, u_{C_k}, w_{\bar{\ell}}, v_{C_k}, v_{C_{k-1}}, \dots, v_{C_1}, w_\ell$ alternating with pairs of non-distinguished vertices unique to B_ℓ . Let F_ℓ be a disk with boundary B_ℓ . See Figure 3.1.

Let y be a vertex contained in cycles $B_{\ell_1}, \dots, B_{\ell_j}$ and let e_i, e_i' be those edges of B_{ℓ_i} containing y , for $i = 1, \dots, j$. These cycles are merged at y by identifying edges $e_1 = e_2'$, $e_2 = e_3'$, \dots , $e_{k-1} = e_k'$, and $e_k = e_1'$. See Figure 3.2.

Let G_0 be the graph derived from edge-disjoint cycles $\{B_\ell \mid \ell \in L\}$ by merging at each vertex w_ℓ for all $\ell \in L$, and merging at each vertex u_c and at each v_c for all $c \in Q$. Note that these operations have the effect of "gluing together" the boundaries of the disks $\{F_\ell \mid \ell \in L\}$. Thus there is a closed, oriented 2-manifold M containing G_0 with identity mapping $I_0: G_0 \rightarrow G_0$; such that for each literal $\ell \in L$, F_ℓ is a face of I_0 .

Let G be the graph derived from G_0 by adding edges

$$E_1 = \left\{ \{w_x, w_{\bar{x}}\} \mid x \in S \right\} \cup \left\{ \{u_c, v_c\} \mid c \in Q \right\}$$

See Figure 3.3.

Lemma 3.1 There exists an imbedding $I: G \rightarrow M$ extending I_0 if and only if Q is satisfiable.

Proof

(\Rightarrow) Suppose I is an imbedding of G into M with the points of G_0 fixed. By construction of G , either $I(\{w_\ell, w_{\bar{\ell}}\}) \subseteq F_\ell$ or $I(\{w_\ell, w_{\bar{\ell}}\}) \subseteq F_{\bar{\ell}}$.

Let $\tau(\ell) = 1$ if $I(\{w_\ell, w_{\bar{\ell}}\}) \subseteq F_{\bar{\ell}}$ and let $\tau(\ell) = 0$ else. We claim $\tau(Q) = 1$. Consider any $C \in Q$. By construction of G , $I(\{u_c, v_c\}) \subseteq F_\ell$ for some $\ell \in C$. But $\{w_\ell, w_{\bar{\ell}}\}$ and $\{u_c, v_c\}$ cannot both be imbedded into F_ℓ , so $I(\{w_\ell, w_{\bar{\ell}}\}) \subseteq F_{\bar{\ell}}$ and $\tau(\ell) = 1$. Thus for each $C \in Q$, we have shown $\tau(C) = 1$.

(\Leftarrow) Suppose τ is a truth assignment with $\tau(BF) = 1$. Let $I: G \rightarrow M$ so that

- (i) the points of G_0 are fixed,
- (ii) for each $\ell \in L$, imbed $\{w_\ell, w_{\bar{\ell}}\}$ into face $F_{\bar{\ell}}$ if $\tau(\ell) = 1$ and into F_ℓ if $\tau(\ell) = 0$.
- (iii) For each $C \in Q$, imbed $\{u_c, v_c\}$ into some face F_ℓ such that $\ell \in C$ and $\tau(\ell) = 1$.

Thus I is the required imbedding of G into M . \square

Since by Theorem 2.2.1 the imbedding extension problem is in NP and it is known [Cook, 1971] that the problem of testing for satisfiability of boolean formulas in CNF form is NP-hard, we have;

Theorem 3.1: The imbedding extension problem is NP-complete.

Recall that an imbedding is quasiplanar if each closed face is homeomorphic to a disk. A slight modification of the above construction yields:

Theorem 3.2: The imbedding extension problem is NP-complete even if the given partial imbedding is quasiplanar.

Proof

Each closed face F_ℓ associated with literal is homeomorphic to a disk. However, observe that certain open faces of I_0 are not associated with literals, and these will be called induced faces. If F is an induced face, then the boundary of F can have no repeated nondistinguished vertices, but may have repeated distinguished vertices--in which case \bar{F} is not homeomorphic to a disk.

For each subsequence $\{y_1, y_2\}$, $\{y_2, y_3\}$ of the boundary of an induced face F , such that y_1, y_3 are distinct nondistinguished vertices and y_2 is a distinguished repeated vertex, add a new edge $\{y_1, y_3\}$ and imbed this edge into F . This has the effect of "closing off" any repeated vertex y_2 . The resulting graph G'_0 has identity mapping $I'_0 : G'_0 \rightarrow G'_0$ which is a quasiplanar imbedding into M .

Let G' be the graph G'_0 plus the edges E_1 . The edges of E_1 can only be imbedded into the faces of $\{F_\ell | \ell \in L\}$. Hence G' has an imbedding extending I'_0 if and only if G has an imbedding extending I_0 if and only if Q is satisfiable. \square

Observe that if the boolean formula Q is in 2-CNF (with but 2 literals per clause) then the constructed graph G' is cubic. Furthermore, this construction requires only log-space. Since 2-SAT is NL-complete, we have:

Corollary 3.1 : The problem of testing if a cubic graph has no imbedding extending a quasiplanar imbedding is NL-hard.

Since the reduction of Theorem 3.2 is parsimonious, we have:

Corollary 3.2 : The problem of counting all nonisotopic imbeddings of a cubic graph extending a given quasiplanar imbedding is #P-complete.

Next we demonstrate that

Theorem 3.3 : The imbedding extension problem is NP-complete for cubic graphs.

Proof

It was shown in [Cook, 1971] that 3-SAT is NP-complete. Let Q be a boolean formula in 3-CNF with set of literals L .

For each literal $\ell \in L$, with $c(\ell) = \{c_1, \dots, c_k\}$, let B'_ℓ be the simple cycle containing distinguished vertices in order

$$w_\ell, u_{c_1, \ell}, \dots, u_{c_k, \ell}, w_{\bar{\ell}}, v_{c_k, \ell}, v_{c_{k-1}, \ell}, \dots, v_{c_1, \ell}, w_\ell$$

alternating with pairs of nondistinguished vertices unique to B'_ℓ .

Also, for each clause $c \in Q$ with $c = \{\ell_1, \ell_2, \ell_3\}$ let y_c, z_c be distinguished vertices and let B_c be the cycle containing distinguished vertices in order

$$y_c, u_{c, \ell_1}, z_c, u_{c, \ell_2}, u_{c, \ell_3}, z_c, v_{c, \ell_3}, y_c, v_{c, \ell_2}, v_{c, \ell_1}, y_c$$

alternating with pairs of nondistinguished vertices. See Figure 3.4.

Let H_0 be the graph derived from edge-disjoint cycles

$\{B'_\ell \mid \ell \in L\} \cup \{B_c \mid c \in BF\}$, by merging at vertices w_ℓ , $u_{c, \ell}$, and $v_{c, \ell}$ for each literal $\ell \in L$ and clauses $c \in C(\ell)$.

As in the previous construction, H_0 has an imbedding J_0 into oriented surface M such that for each $\ell \in L$, there is a face F'_ℓ with boundary B'_ℓ , and for each clause $c \in Q$, there is a closed face F_c with boundary B_c .

Let H be the graph H_0 plus edges $\left\{ \{w_\ell, w_{\bar{\ell}}\} \mid \ell \in L \right\} \cup \left\{ \{u_{c,\ell}, v_{c,\ell}\} \mid \ell \in L, c \in C(\ell) \right\} \cup \left\{ \{y_c, z_c\} \mid c \in Q \right\}$.

Lemma 3.2 There exists an imbedding $J : H \rightarrow M'$ extending J_0 if and only if Q is satisfiable.

Proof

(\Rightarrow) Given an imbedding $J : H \rightarrow M'$ extending J_0 , let $\tau(\ell) = 1$ if $J(\{w_\ell, w_{\bar{\ell}}\}) \subseteq F'_\ell$ and let $\tau(\ell) = 0$ else.

If c is a clause of Q , then since it is impossible to simultaneously imbed all the edges $\left\{ \{u_{c,\ell}, v_{c,\ell}\} \mid \ell \in C \right\} \cup \{y_c, z_c\}$ into F_c , it follows that $J(\{u_{c,\ell}, v_{c,\ell}\}) \subseteq F'_\ell$ for some $\ell \in C$. Hence $J(\{w_\ell, w_{\bar{\ell}}\}) \subseteq F'_\ell$ and $\tau(c) = \tau(\ell) = 1$.

(\Leftarrow) Given a truth assignment τ satisfying Q , we construct an imbedding $J : H \rightarrow M$ as follows:

- (i) let the points of H_0 be fixed J_0 .
- (ii) for each literal $\ell \in L$, let $J(\{w_\ell, w_{\bar{\ell}}\}) \subseteq F'_\ell$ if $\tau(\ell) = 1$
 $\subseteq F_\ell$ if $\tau(\ell) = 0$
- (iii) for each clause $c \in Q$ where $c = \{\ell_1, \ell_2, \ell_3\}$ with $\tau(\ell_1) = 1$,
 let $J(\{u_{c,\ell_1}, v_{c,\ell_1}\}) \subseteq F'_{\ell_1}$, and
 let $J(\{u_{c,\ell_2}, v_{c,\ell_2}\}, \{u_{c,\ell_3}, v_{c,\ell_3}\}) \subseteq F_c$.

It is easy to show that there is always an imbedding

$J(\{y_c, z_c\}) \subseteq F_c$. \square

4. Imbedding Extension Problems Polynomial-Time Reducible to 2-SAT

In the previous section, the imbedding extension problem for graph G and partial imbedding I_0 was shown to be NP-complete in the case either

- (1) G is cubic, or
- (2) I_0 is quasiplanar.

Here we show that if (1) and (2) both hold, then there is a polynomial-time algorithm for the imbedding extension problem.

Fix a graph $G = (V, E)$, a subgraph $G_0 = (V_0, E_0)$, and imbedding $I_0 : G_0 \rightarrow M$. Let G' be the graph derived from G by

- (1) deleting all edges of E_0
- (2) substituting for each edge $\{u, v\} \in E - E_0$ with $v \in V_0$, an attachment edge $\{u, uv\}$ with new vertex uv .

Let a piece p be a graph derived from a nontrivial connected component of G' by replacing each vertex uv with the corresponding original vertex v in each attachment vertex $\{u, uv\}$. Note that each piece p is a connected subgraph of G_0 . The attachment vertices of p are those vertices of p contained in V_0 . Furthermore, for any imbedding I of G into M extending I_0 , $I(p)$ is contained in a single closed face of I_0 .

Each attachment vertex of p must be contained on the boundary $B(F)$.

Lemma 4.1: If $\bar{F} = F \cup B(F)$ is homeomorphic to a disk, then $p + B(F)$ is a planar graph.

Proof

Since \bar{F} is homeomorphic to a disk, there is a homeomorphism h from \bar{F} into the plane. Thus $h \cdot I$, restricted to p , is a homeomorphism into the plane. \square

Lemma 4.2: If \bar{F} is homeomorphic to a disk and I_p is an imbedding of $p + B(F)$ into a sphere M_0 , then there exists an imbedding $I' : G \rightarrow M$ such that $I'(G-p) = I(G-p)$ and $I_p(p)$, $I'(p)$ are represented by the same combinatorial imbedding.

Proof

$M_0 - I_p(B(F))$ contains exactly two maximal, connected regions D_1, D_2 each homeomorphic to a disk. Let us assume, without loss of generality, that $I_p(p) \subseteq \bar{D}_1$. Since \bar{F} is homeomorphic to a disk, there is a region $R \subseteq \bar{F}$ homeomorphic to a disk containing $I(p)$ but disjoint from $I(G-p)$. Since both R and \bar{F} are homeomorphic to a disk, there is a homeomorphism $h : R \rightarrow \bar{F}$ such that the points of $B(F)$ are fixed in $H \cdot I_p$. Thus we may let $I'(p) = h(I_p(p))$; $I'(p)$ is contained in R and thus is disjoint from $I'(G-p) = I(G-p)$. \square

For each piece p , let $F(p)$ be the set of those faces F of I_0 such that

- (1) each attachment vertex of p is contained in $B(F)$, and
- (2) $B(F) + P$ is planar.

Pieces p_1, p_2 interfere at face $F \in F(p_1) \cap F(p_2)$ if there is no imbedding from $p_1 + p_2$ into \bar{F} in which the attachment vertices are fixed. See Figure 4.1.

Lemma 4.3: If \bar{F} is homeomorphic to a disk, pieces p_1, p_2 interfere at face $F \in F(p_1) \cap F(p_2)$ just in the case there are distinct vertices u_1, u_2, v_1, v_2 occurring in this order in $B(F)$ such that u_i, v_i are attachment vertices of p_i , for $i = 1, 2$.

Proof

If such vertices u_1, u_2, v_1, v_2 exist, then there is a path q_i in p_i from u_i to v_i , for $i = 1, 2$. It is clear that there is no imbedding from $q_1 + q_2$ into \bar{F} , and hence there is no imbedding from $p_1 + p_2$ into \bar{F} .

On the other hand, if such vertices u_1, u_2, v_1, v_2 do not exist, then \bar{F} can be partitioned into a pair of regions D_1 and D_2 , each homeomorphic to a disk, with all attachment vertices of p_i contained in D_i , for $i = 1, 2$ and such that $D_1 \cap D_2$ contains at most two points of $B(F)$. But since $B(F) + p_1$ and $B(F) + p_2$ are both planar, there are imbeddings $I_1 : p_1 \rightarrow D_1$ and $I_2 : p_2 \rightarrow D_2$ which combined form an imbedding from $p_1 + p_2$ into $\bar{F} = D_1 \cup D_2$. \square

Now, let us assume

- (1) I_0 is quasiplanar, and
- (2) for each piece p , $|F(p)| \leq 2$.

As a consequence of Lemma 4.3, to find an imbedding of G extending I_0 we need only find an assignment of each piece p of G to a face F of I_0 such that $B(F) + p$ is planar, and no pair of pieces interfere relative to this face assignment.

For each piece p and face $F \in F(p)$, let there be a boolean variable $x_{p,F}$. Let Q be a set of clauses containing

- (1) for each piece p ,
 - (a) a clause $(x_{p,F})$ if $F(p) = \{F\}$
 - (b) else if $F(p) = \{F_1, F_2\}$, a pair of clauses $(x_{p,F_1} \vee x_{p,F_2})$ and $(\bar{x}_{p,F_1} \vee \bar{x}_{p,F_2})$.
- (2) a clause $(\bar{x}_{p_1,F} \vee \bar{x}_{p_2,F})$ for each pair of pieces p_1, p_2 conflicting relative to face $F \in F(p_1) \cap F(p_2)$.

For example, see Figure 3.2.

Lemma 4.4: Q is satisfiable if and only if there exists an imbedding of G extending I_0 .

Proof

Let τ be a truth assignment satisfying Q . Imbed each piece p into some face $F \in F(p)$ such that $\tau(X_{p,F}) = 1$. Note that no pair of faces p_1, p_2 which conflict are thus imbedded into the same face F , since $(\bar{X}_{p_1,F} \vee \bar{X}_{p_2,F}) \in Q$.

Conversely, suppose I is an imbedding from G into M extending I_0 . Let $\tau(X_{p,F}) = 1$ just in the case $I(p) \subseteq \bar{F}$ for each piece p and face $F \in F(p)$. This truth assignment τ may easily be shown to satisfy Q . \square

Observe that if $G = (V, E)$, then the formula Q is of size $O(|V|^2 + |E|)$. Using the results of [Jones, Lien, and Lausar, 1976] and the efficient graph processing techniques of [Tarjan, 1975], we determine the satisfiability of a formula in 2-CNF in linear time. Also, the [Hopcroft and Tarjan, 1974] planar graph recognition algorithm runs in linear time. Thus, we have established

Theorem 4.1: The imbedding extension problem for graph $G = (V, E)$ and quasiplanar partial imbedding I_0 in which $|F(p)| \leq 2$ for each piece p , can be solved in time $O(|V|^2 + |E|)$.

If graph G is cubic and I_0 is quasiplanar, then it follows that $|F(p)| \leq 2$ for each piece p .

Corollary 4.1: The imbedding extension problem for a cubic graph and quasiplanar partial imbedding can be solved in quadratic time.

This result was also independently discovered by [Miller, 1978].

5. Conclusion

While this paper has demonstrated that the imbedding extension problem is NP-complete, the complexity of the imbedding problem for arbitrary surfaces remains open.

A possible line of attack would be to demonstrate that given an imbedding oracle for graph G , which provides for free an imbedding of G into an oriented surface of minimal genus, a known NP-complete problem for G can be solved in polynomial time.

It is encouraging that certain known NP-complete problems, such as MAX-CUT and FEEDBACK-ARCS, are polynomial-time for planar graphs. Perhaps these results will generalize to graphs with imbeddings into fixed surfaces.

As a corollary to our NP-completeness results, we have shown that counting all nonisotopic imbeddings of a graph G extending a given partial imbedding I_0 is #P-complete. However, the complexity of counting nonisotopic imbeddings of a graph G into a surface M is open.

All known [Hopcroft and Tarjan, 1973; Fontet, 1976; Colbourn and Booth, 1977] efficient algorithms for testing isomorphism and computing the automorphism partitioning of planar graphs rely on the (apparently crucial) fact that the 3-connected components of a planar graph have a unique (up to isotopy) imbedding into the sphere. A characterization of those graphs rigid on surface M (i.e., with a unique imbedding into surface M , up to isotopy) would be of some interest. Given an imbedding oracle, isomorphism and automorphism partitioning of rigid graphs is polynomial-time. Perhaps there is a relationship between recognition problems for rigid graphs and for formulas in CNF with unique satisfying instances.

Postscript

Gary Miller has also shown the imbedding extension problem NP-complete, although his proof does not tighten to the restricted cases of cubic graphs or quasiplanar imbeddings. His proof uses a complex reduction from the coloring problem for circular arc graphs, recently shown NP-complete [Papadimitriou, 1978].

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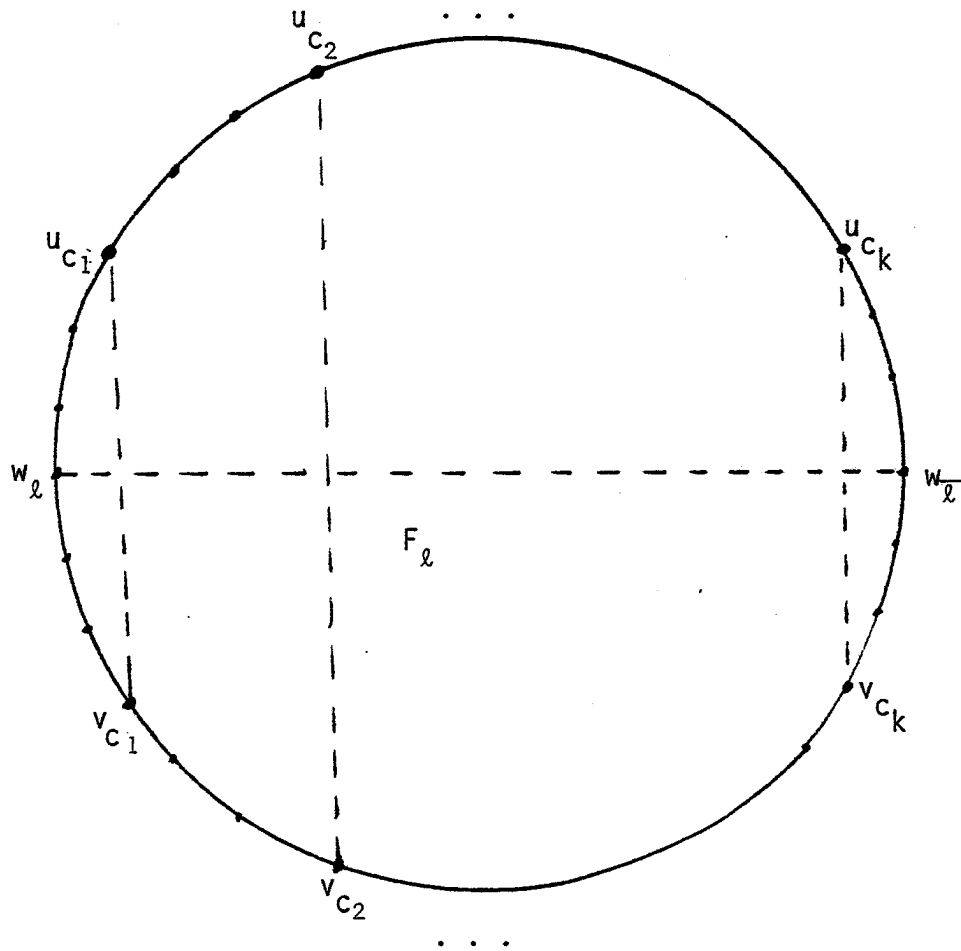


Figure 3.1: Face F_l , where c_1, c_2, \dots, c_k are the clauses containing literal l .

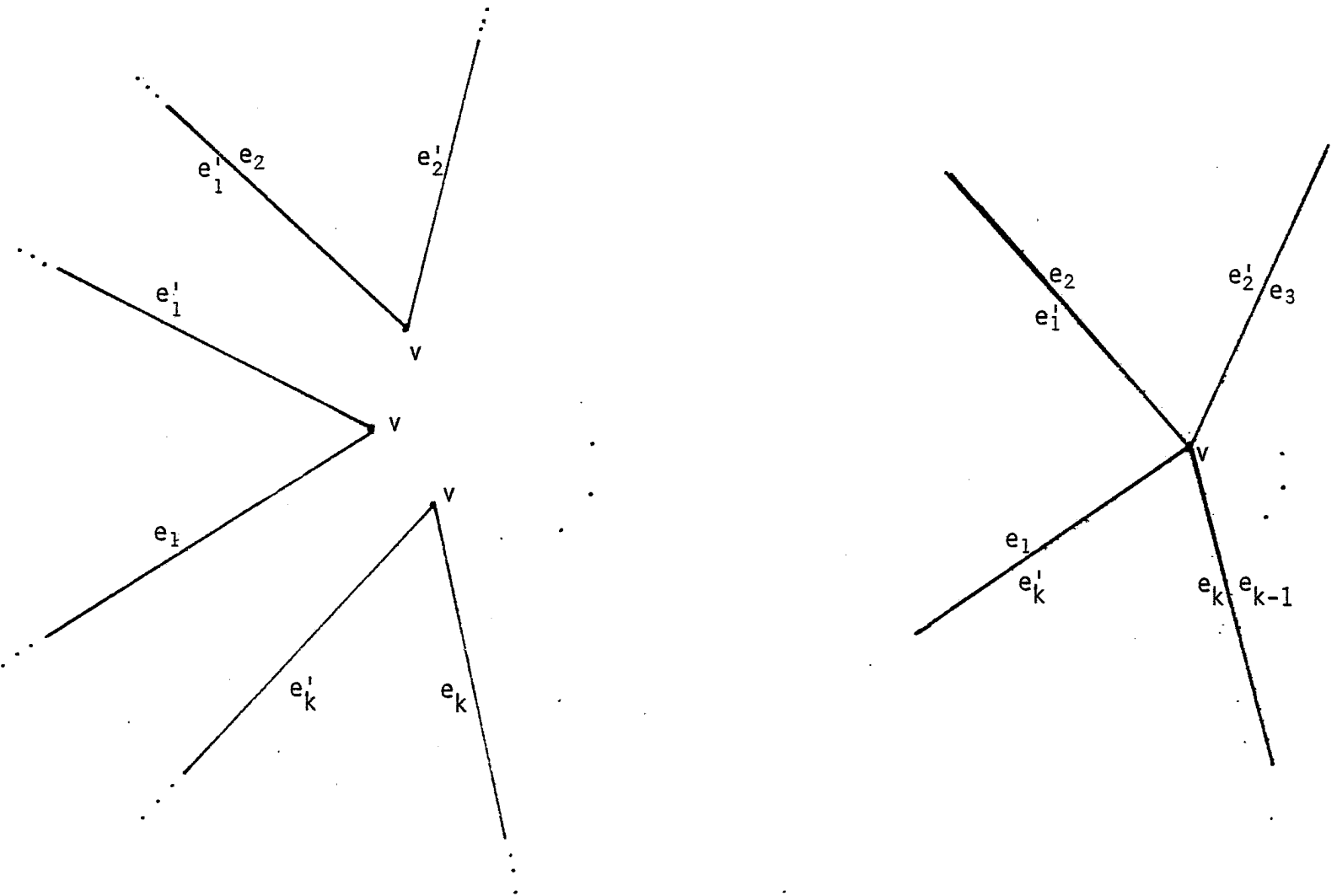


Figure 3.2: A merge at vertex v .

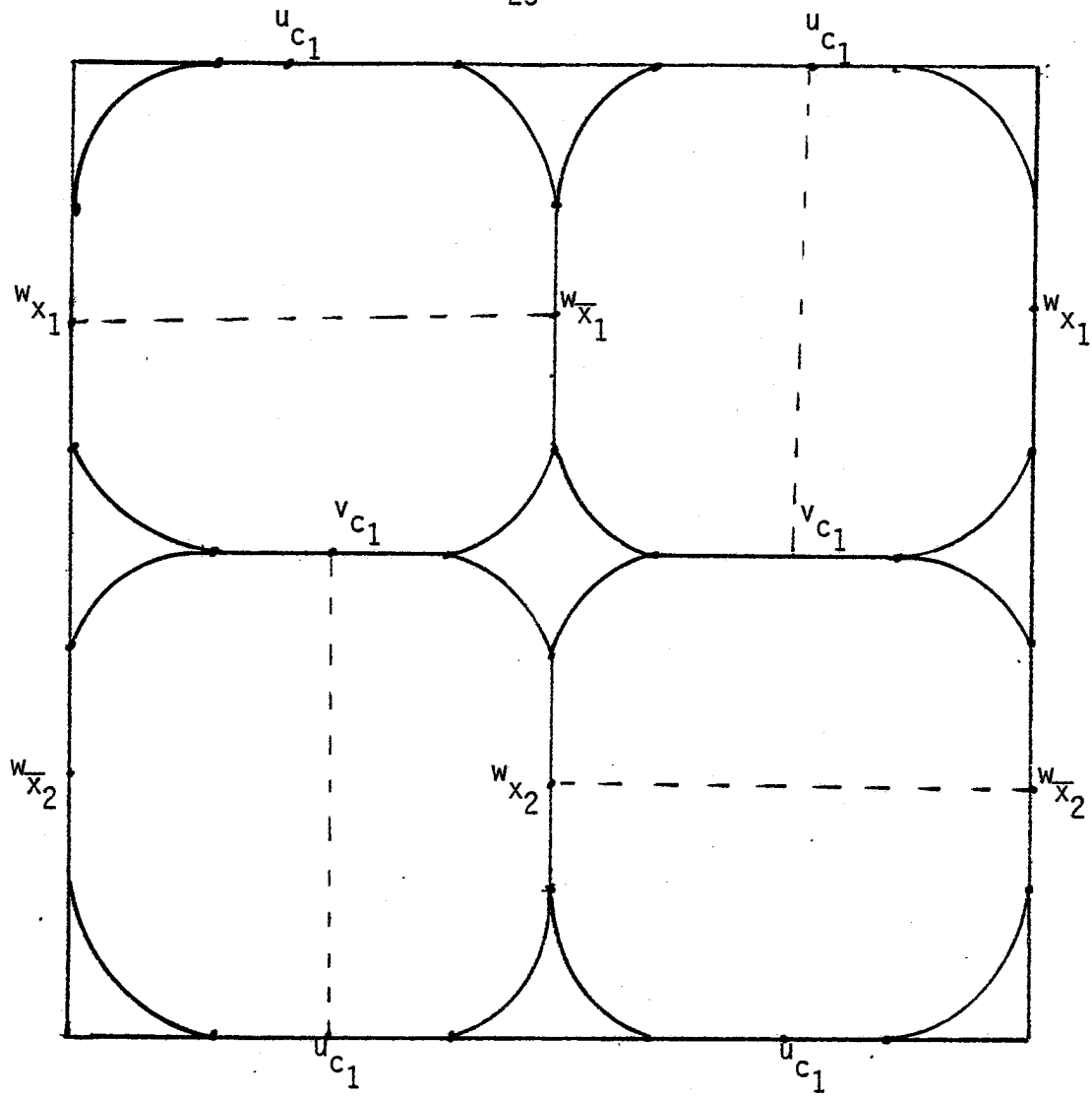


Figure 3.3: Partially imbedded graph G_F constructed from boolean formula $F = c_1 \wedge c_2$ where $c_1 = (x_1 \vee \bar{x}_2)$ and $c_2 = (\bar{x}_1 \vee x_2)$.

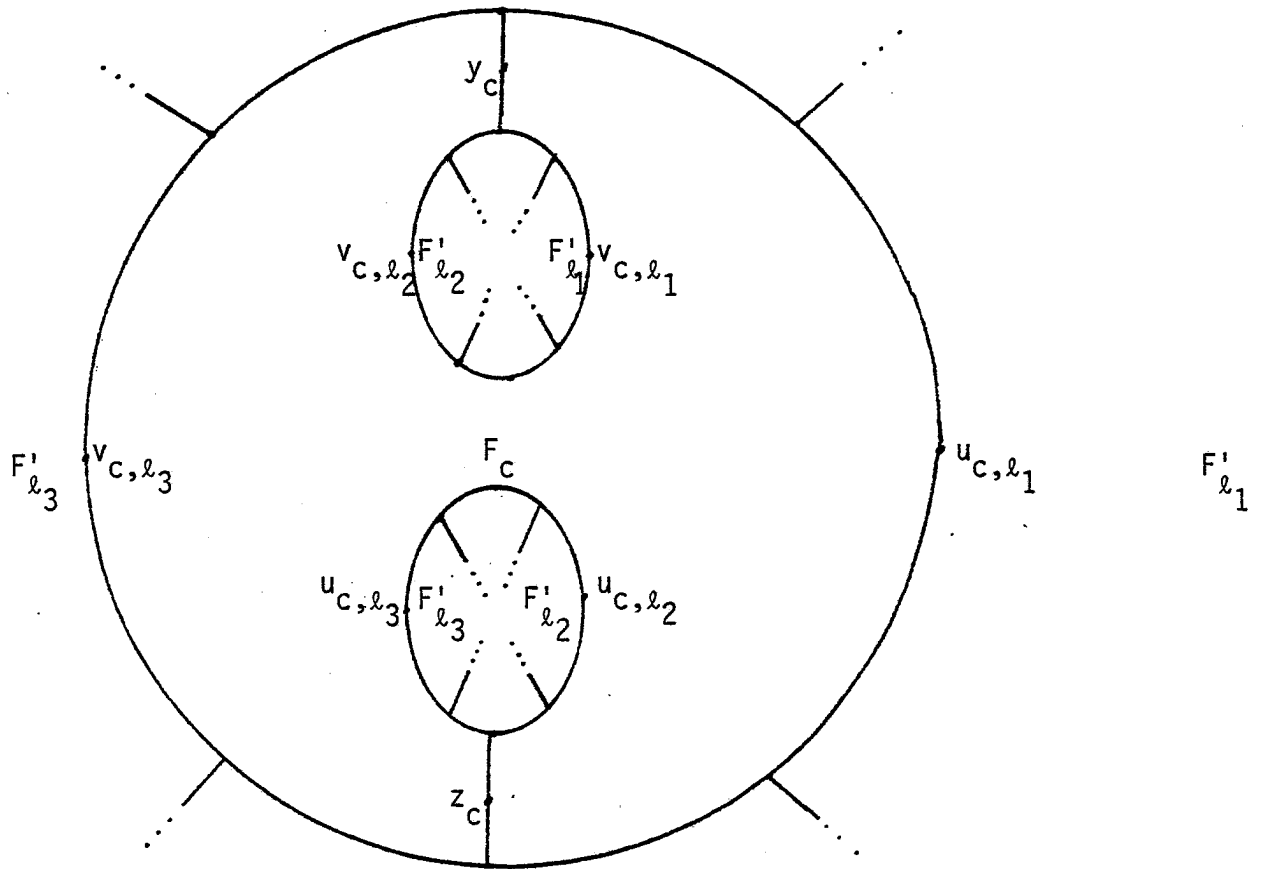


Figure 3.4: The face F_c associated with clause c . Note that edges $\left\{ \{u_{c,l_i}, v_{c,l_i}\} \mid i=1,2,3 \right\} \cup \{y_c, z_c\}$ cannot all be simultaneously imbedded into F_c .